`scriptHelmholtz8bie.m`

Behavior of error for eight B.I.E. for the Helmholtz equation

Prepared by: Andrea Carosso & Francisco-Javier Sayas

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The goal of this script is testing errors for different boundary integral formulations of

$$\Delta U + k^2 U = 0 \quad \text{in } \Omega_+, \quad \partial_r U - \imath k U = o(r^{-1/2}) \quad \text{as } r \to \infty,$$

with

$$U = \beta_0 = -U_{\text{inc}} \quad \text{on } \Gamma, \quad \text{or} \quad \partial_\nu U = \beta_1 = -\partial_\nu U_{\text{inc}} \quad \text{on } \Gamma.$$

Physical parameters, exact solution, geometry, and observation points. We will write everything in terms of the resolvent equation

$$\Delta U - s^2 U = 0 \quad s = -\imath k.$$

The exact solution is

$$\begin{aligned}
U(\mathbf{z}) &:= H_0^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{\text{sc}}|)\\
U_x(\mathbf{z}) &:= -\imath\, s\,(z_1 - x_1^0) H_1^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{\text{sc}}|)\frac{1}{|\mathbf{z} - \mathbf{x}^0|}\\
U_y(\mathbf{z}) &:= -\imath\, s\,(z_2 - x_2^0) H_1^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{\text{sc}}|)\frac{1}{|\mathbf{z} - \mathbf{x}^0|}\\
\nabla U(\mathbf{z}) &:= (U_x(\mathbf{z}), U_y(\mathbf{z}))
\end{aligned}$$

where $\mathbf{x}^{\text{sc}} := (0.1, 0.2)$ has to be in the interior of the domain. The incident wave

$$U^{\text{inc}} = -U,$$

gives U as the exact solution of the exterior problem. The geometry is the union of the TV-shape (see deltaBEM documentation), which fits in a square $[-1.6, 1.6]^2$, with the ellipse

$$(x_1 - 4)^2 + \tfrac{1}{4}(x_2 - 5)^2 = 1.$$

The solution will be observed in four exterior points:

$$\mathbf{x}_1^{\text{obs}} := (0, 4), \quad \mathbf{x}_2^{\text{obs}} := (4, 0), \quad \mathbf{x}_3^{\text{obs}} := (-4, 2), \quad \mathbf{x}_4^{\text{obs}} := (2, -4).$$

Discretization. We will use $2N$ points to discretize the TV-shape and N points to discretize the ellipse. The domains are sampled (three times $\varepsilon = 0, -1/6, 1/6$) and the discrete geometries are merged. Next we compute the elements of the Calderón Calculus,

$$\mathrm{Q}, \quad \mathrm{M}, \quad \mathrm{V}(s), \quad \mathrm{K}(s), \quad \mathrm{J}(s), \quad \mathrm{W}(s).$$

The four operators depend on s and are output as function handles. We next sample the incident wave (see Chapter 3 of the deltaBEM documentation)

$$\boldsymbol{\beta}_0, \quad \boldsymbol{\beta}_1.$$

Finally we create the single and double layer potentials

$$\mathrm{S}(s), \quad \mathrm{D}(s)$$

at the observation points.

Integral formulations. Non-physical densities will be denoted $\boldsymbol{\eta}$ for the S.L. potential and $\boldsymbol{\psi}$ for the D.L. potential. The effective density for the D.L. potential (the one that appears in potential expressions, but not on integral equations) is $\mathrm{Q}\boldsymbol{\psi}$. The approximation for the exterior trace and normal derivatives will be $\boldsymbol{\phi}$ and $\boldsymbol{\lambda}$ respectively. The effective approximation of the exterior trace (appearing in potential expressions, and also as the final approximation of the exterior trace) will be $\boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}$.

1. (iD01 - Dirichlet problem – indirect formulation)

$$\mathrm{V}(s)\boldsymbol{\eta} = \boldsymbol{\beta}_0, \quad \mathrm{M}\boldsymbol{\lambda} = -\tfrac{1}{2}\mathrm{M}\boldsymbol{\eta} + \mathrm{J}(s)\boldsymbol{\eta}, \quad U_h = \mathrm{S}(s)\boldsymbol{\eta}, \quad \mathrm{M}\boldsymbol{\phi} = \boldsymbol{\beta}_0, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

2. (iD02 - Dirichlet problem – indirect formulation)

$$\tfrac{1}{2}\mathrm{M}\boldsymbol{\psi} + \mathrm{K}(s)\boldsymbol{\psi} = \boldsymbol{\beta}_0, \quad \mathrm{M}\boldsymbol{\lambda} = -\mathrm{W}(s)\boldsymbol{\psi}, \quad U_h = \mathrm{D}(s)\mathrm{Q}\boldsymbol{\psi}, \quad \mathrm{M}\boldsymbol{\phi} = \boldsymbol{\beta}_0, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

3. (dD01 - Dirichlet problem – direct formulation)

$$\mathrm{V}(s)\boldsymbol{\lambda} = -\tfrac{1}{2}\mathrm{M}\boldsymbol{\phi} + \mathrm{K}(s)\boldsymbol{\phi}, \quad U_h = \mathrm{D}(s)\mathrm{Q}\boldsymbol{\phi} - \mathrm{S}(s)\boldsymbol{\lambda}, \quad \mathrm{M}\boldsymbol{\phi} = \boldsymbol{\beta}_0, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

4. (dD02 - Dirichlet problem – direct formulation)

$$\tfrac{1}{2}\mathrm{M}\boldsymbol{\lambda} + \mathrm{J}(s)\boldsymbol{\lambda} = -\mathrm{W}(s)\boldsymbol{\phi}, \quad U_h = \mathrm{D}\mathrm{Q}\boldsymbol{\phi} - \mathrm{S}(s)\boldsymbol{\lambda}, \quad \mathrm{M}\boldsymbol{\phi} = \boldsymbol{\beta}_0, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

5. (iN01 - Neumann problem – indirect formulation)

$$-\tfrac{1}{2}\mathrm{M}\boldsymbol{\eta} + \mathrm{J}(s)\boldsymbol{\eta} = \boldsymbol{\beta}_1, \quad \mathrm{M}\boldsymbol{\phi} = \mathrm{V}(s)\boldsymbol{\eta}, \quad U_h = \mathrm{S}(s)\boldsymbol{\eta}, \quad \mathrm{M}\boldsymbol{\lambda} = \boldsymbol{\beta}_1, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

6. (iN02 - Neumann problem – indirect formulation)

$$\mathrm{W}(s)\boldsymbol{\psi} = -\boldsymbol{\beta}_1, \quad \mathrm{M}\phi = \tfrac{1}{2}\mathrm{M}\boldsymbol{\psi} + \mathrm{K}(s)\boldsymbol{\psi}, \quad U_h = \mathrm{D}(s)\mathrm{Q}\boldsymbol{\psi}, \quad \mathrm{M}\boldsymbol{\lambda} = \boldsymbol{\beta}_1, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

7. (dN01 - Neumann problem – direct formulation)

$$-\tfrac{1}{2}\mathrm{M}\boldsymbol{\phi} + \mathrm{K}(s)\boldsymbol{\phi} = \mathrm{V}(s)\boldsymbol{\lambda}, \quad U_h = \mathrm{D}(s)\mathrm{Q}\boldsymbol{\phi} - \mathrm{S}(s)\boldsymbol{\lambda}, \quad \mathrm{M}\boldsymbol{\lambda} = \boldsymbol{\beta}_1, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

8. (dN02 - Neumann problem – direct formulation)

$$\mathrm{W}(s)\boldsymbol{\phi} = -\tfrac{1}{2}\mathrm{M}\boldsymbol{\lambda} - \mathrm{J}(s)\boldsymbol{\lambda}, \quad U_h = \mathrm{D}(s)\mathrm{Q}\boldsymbol{\phi} - \mathrm{S}(s)\boldsymbol{\lambda}, \quad \mathrm{M}\boldsymbol{\lambda} = \boldsymbol{\beta}_1, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

Computation of errors. In all the formulations we will have approximations

$$U_h(\mathbf{x}_i^{\text{obs}}) \approx U(\mathbf{x}_i^{\text{obs}}), \quad i = 1, \dots, 4, \quad \varphi_j \approx U(\mathbf{m}_i) =: \beta_{0,i}^{\text{ex}}, \quad \lambda_j \approx \nabla U(\mathbf{m}_i) \cdot \mathbf{n}_i =: \beta_{1,i}^{\text{ex}}.$$

Note that $|\mathbf{n}_i| = \mathcal{O}(N^{-1})$. We then compute errors:

$$\begin{aligned} e_{\text{Pot}} &:= \max |U_h(\mathbf{x}_i^{\text{obs}}) - U(\mathbf{x}_i^{\text{obs}})| = \mathcal{O}(N^{-3}), \\ e_\lambda &:= N \max |\lambda_i - \beta_{1,i}^{\text{ex}}| = \mathcal{O}(N^{-3}). \\ e_\varphi &:= \max |\varphi_i - \beta_{0,i}^{\text{ex}}| = \mathcal{O}(N^{-3}), \end{aligned}$$

The errors are output as a row vector (in the given order).
There are two different modes to run this script:

- If a variable `external` HAS NOT been defined, the script requests the user to input: k, N, the experiment number (1 to 8) and whether the fork will be used or not.
- If the variable `external` has been defined, these data need to have been declared in advance. In this case, the errors are appended as an additional row to a matrix of errors.