`scriptHowToUseBDFCQ.m`

Solve the wave equation with BDFCQ

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This script is a demonstration of how to use deltaBEM and BDF2 based CQ to solve four different formulations of the wave equation. We compute errors to show the convergence of the method (at the optimal order of two, because of the underlying BDF time stepper).

Problem parameters. In all following examples, Ω will be an ellipse centered at the origin with the x semiaxis having length one and the y semiaxis having length three. We let Γ denote its boundary. This is discretized in space with N points and in time with M time steps. Data for the problem will be a plane wave traveling in the direction $\mathbf{d} = (1/\sqrt{2}, 1/\sqrt{2})$ and transmitting the signal

$$f(t) = \sin(2t)^5 \chi(t \geq 0).$$

Computation of errors. To compute errors, we construct the various potential representations at the three interior observation points

$$\mathbf{x}_1^{obs} = (0.1, 0.2), \quad \mathbf{x}_2^{obs} = (-0.1, -0.3), \quad \mathbf{x}_3^{obs} = (0.5, 0).$$

One of the hypotheses of CQ time discretization is that the incident wave not interact with the scatterer at $T = 0$. To enforce this, we can shift the signal with a time lag to make sure we do not have scattering beginning at $T = 0$. This parameter is passed by the user as input. Because of the oblong shape we are testing, a time lag of about 2 seconds is necessary. We know the exact solution in the interior is a planewave, so we use this to compute errors at the observation points at the final time T:

$$\text{error} = \max_j \left| u(\mathbf{x}^{obs}, T) - u^h(\mathbf{x}_j^{obs}, T) \right|,$$

where we denote $u(\mathbf{x}^{obs}, T)$ the exact solution of the final time on the observation points and $u^h(\mathbf{x}_j^{obs}, T)$ the discrete computed solution on the observation points.

Experiment 1 The first experiment is an interior Dirichlet problem with a single layer ansatz. We denote $\beta_0 = \gamma u^{inc}$, and solve the PDE

$$\begin{aligned} \Delta u &= u_{tt} && \text{in } \Omega \times [0, T] \\ \gamma u &= \beta_0 && \text{on } \Gamma \times [0, T]. \end{aligned}$$

Our use of a single layer ansatz for the solution leads to the time domain boundary integral equation

$$\mathcal{V} * \boldsymbol{\eta} = \boldsymbol{\beta}_0$$

where we solve for the unknown (and non-physical) density $\boldsymbol{\eta}$ and then postprocess to compute the solution

$$u = \mathcal{S} * \boldsymbol{\eta}.$$

Experiment 2 The second experiment is an indirect formulation for the Neumann problem. Setting $\beta_1 = \partial_\nu u^{inc}$, we seek a solution to the PDE

$$\begin{aligned} \Delta u &= u_{tt} && \text{in } \Omega \times [0, T] \\ \partial_\nu u &= \beta_1 && \text{on } \Gamma \times [0, T]. \end{aligned}$$

We make a double layer ansatz for the solution, so we must solve the integral equation on $\Gamma \times [0, T]$:

$$-\mathcal{W} * \boldsymbol{\psi} = \boldsymbol{\beta}_1, \tag{1}$$

followed by the postprocessing

$$u = \mathcal{D} * Q\boldsymbol{\psi}.$$

Here, $\mathcal{D}$ acts on the effective density $Q\psi$, where Q is a quadrature-related matrix. As before, we solve the BIE (1) with deltaBEM in space and BDF2 based CQ in time, and postprocess the solution on the interior observation points with the same.

Experiment 3 This experiment is a direct formulation for the Dirichlet problem. We take as the unknown $\boldsymbol{\lambda} := \partial_\nu u^{inc}$ and represent the solution to the interior problem with Kirchhoff's formula:

$$u = \mathcal{S} * \boldsymbol{\lambda} - \mathcal{D} * \boldsymbol{\beta}_0.$$

When we take the trace of the above equation and rearrange the terms, we need to solve the BIE

$$\mathcal{V} * \boldsymbol{\lambda} = \frac{1}{2}\mathbf{M}\boldsymbol{\beta}_0 + \mathcal{K} * \boldsymbol{\beta}_0.$$

The matrix $\mathbf{M}$ is a mass matrix that is part of the discrete formulation (on the continuous level, this is just the identity operator).

Experiment 4 The last experiment in this script is solving the Neumann problem for the wave equation with a Direct formulation. Our unknown will be $\phi := \gamma u^{inc}$. The BIE we have to solve for ϕ is

$$-\mathcal{W} * \boldsymbol{\phi} = -\frac{1}{2}\mathbf{M}\boldsymbol{\beta}_1 + \mathcal{J} * \boldsymbol{\beta}_1,$$

followed by the postprocessing

$$u = \mathcal{S} * \boldsymbol{\lambda} - D * Q\boldsymbol{\phi}.$$

There are two different modes to run this script. If a variable external HAS NOT been defined, the script requests the user to input: N (spatial discretization parameter), M (time discretization parameter), T (the final time), the time lag, the experiment number (1, 2, 3, or 4) and whether the fork will be used or not. If the variable external has been defined, these data need to have been declared in advance. In this case, the errors are appended as an additional row to a matrix of errors.