`scriptLaplace2bie.m`

Behavior of error for two B.I.E. for the Laplace equation

Prepared by: Andrea Carosso & Francisco-Javier Sayas
Last modified: January 12, 2015

The goal of this script is testing errors for different boundary integral formulations of

$$\Delta u = 0 \qquad \text{in } \Omega_+, \qquad u = u_\infty + o(r^{-1}) \qquad \text{as } r \to \infty,$$

with

$$u = \beta_0 \qquad \text{on } \Gamma, \qquad \text{or} \qquad \partial_\nu u = \beta_1 \qquad \text{on } \Gamma.$$

Exact solution, geometry, and observation points. The exact solution is

$$\begin{aligned} u(\mathbf{z}) &:= \frac{z_1}{|\mathbf{z}|^2} \\ u_x(\mathbf{z}) &:= \frac{1}{|\mathbf{z}|^2} - \frac{2z_1^2}{|\mathbf{z}|^4} \\ u_y(\mathbf{z}) &:= \frac{1}{|\mathbf{z}|^2} - \frac{2z_2^2}{|\mathbf{z}|^4} \\ \nabla u(\mathbf{z}) &:= (u_x(\mathbf{z}), u_y(\mathbf{z})) \end{aligned}$$

The geometry is the union of the TV-shape (see deltaBEM documentation), which fits in a square $[-1.6, 1.6]^2$, with the ellipse

$$(x_1 - 4)^2 + \tfrac{1}{4}(x_2 - 5)^2 = 1.$$

The solution will be observed at four exterior points:

$$\mathbf{x}_1^{\text{obs}} := (0, 4), \quad \mathbf{x}_2^{\text{obs}} := (4, 0), \quad \mathbf{x}_3^{\text{obs}} := (-4, 2), \quad \mathbf{x}_4^{\text{obs}} := (2, -4).$$

Discretization. We will use $2N$ points to discretize the TV-shape and N points to discretize the ellipse. The domains are sampled three times $\varepsilon = 0, -1/6, 1/6$ and the discrete geometries are merged. Next we compute the elements of the Calderón Calculus,

$$\text{Q}, \quad \text{M}, \quad \text{V}, \quad \text{K}, \quad \text{J}, \quad \text{W}, \quad \text{C}.$$

We next sample the exact solution on the boundary,

$$\beta_0, \qquad \beta_1.$$

Finally, we create the single and double layer potentials

$$\text{S}, \qquad \text{D}$$

at the observation points.

Integral formulations. Non-physical densities will be denoted $\boldsymbol{\eta}$ for the S.L. potential and $\boldsymbol{\psi}$ for the D.L. potential. The effective density for the D.L. potential (the one that appears in potential expressions, but not on integral equations) is $\mathrm{Q}\boldsymbol{\psi}$. The approximation for the exterior trace and normal derivatives will be $\boldsymbol{\phi}$ and $\boldsymbol{\lambda}$ respectively. The effective approximation of the exterior trace (appearing in potential expressions, and also as the final approximation of the exterior trace) will be $\boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}$.

1. (iD01 - Dirichlet problem – indirect formulation)

$$\begin{pmatrix} \mathrm{V} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix} \boldsymbol{\eta} = \boldsymbol{\beta}_0, \quad \mathrm{M}\boldsymbol{\lambda} = -\tfrac{1}{2}\mathrm{M}\boldsymbol{\eta} + \boldsymbol{\eta}, \quad u_h = \mathrm{S}\boldsymbol{\eta}, \quad \mathrm{M}\boldsymbol{\phi} = \boldsymbol{\beta}_0, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

2. (iN02 - Neumann problem – indirect formulation)

$$\mathrm{W}\boldsymbol{\psi} + \mathrm{C}\boldsymbol{\psi} = -\boldsymbol{\beta}_1, \quad \mathrm{M}\phi = \tfrac{1}{2}\mathrm{M}\boldsymbol{\psi} + \mathrm{K}\boldsymbol{\psi}, \quad u_h = \mathrm{DQ}\boldsymbol{\psi}, \quad \mathrm{M}\boldsymbol{\lambda} = \boldsymbol{\beta}_1, \quad \boldsymbol{\varphi} = \mathrm{Q}\boldsymbol{\phi}.$$

Computation of errors. In all the formulations we will have approximations

$$u_h(\mathbf{x}_i^{\mathrm{obs}}) \approx u(\mathbf{x}_i^{\mathrm{obs}}), \quad i = 1, \ldots, 4, \qquad \varphi_j \approx u(\mathbf{m}_i) =: \beta_{0,i}^{\mathrm{ex}}, \qquad \lambda_j \approx \nabla u(\mathbf{m}_i) \cdot \mathbf{n}_i =: \beta_{1,i}^{\mathrm{ex}}.$$

Note that $|\mathbf{n}_i| = \mathcal{O}(N^{-1})$. We then compute errors:

$$\begin{aligned} e_{\mathrm{Pot}} &:= \max |u_h(\mathbf{x}_i^{\mathrm{obs}}) - u(\mathbf{x}_i^{\mathrm{obs}})| = \mathcal{O}(N^{-3}), \\ e_{\lambda} &:= N \max |\lambda_i - \beta_{1,i}^{\mathrm{ex}}| = \mathcal{O}(N^{-3}). \\ e_{\varphi} &:= \max |\varphi_i - \beta_{0,i}^{\mathrm{ex}}| = \mathcal{O}(N^{-3}), \end{aligned}$$

The errors are output as a row vector (in the given order).

There are two different modes to run this script. If a variable `external` HAS NOT been defined, the script requests the user to input: N, the experiment number (1 or 2) and whether the fork will be used or not. If the variable `external` has been defined, these data need to have been declared in advance. In this case, the errors are appended as an additional row to a matrix of errors.