`scriptMixedBVP.m`

Scattering from two obstacles with different boundary conditions.

Prepared by: Andrea Carosso & Francisco-Javier Sayas
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The goal of this script is to simulate scattering from two obstacles, one with a Dirichlet boundary condition, the other with a Neumann boundary condition, and to then plot the solution and compute the errors. We solve the problem

$$\Delta U + k^2 U = 0 \qquad \text{in } \Omega_+, \qquad \partial_r U - \imath k U = o(r^{-1/2}) \quad \text{as } r \to \infty,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$, with boundary conditions

$$U = \beta_0 = -U_{\text{inc}} \quad \text{on } \Gamma_1, \qquad \text{and} \qquad \partial_\nu U = \beta_1 = -\partial_\nu U_{\text{inc}} \quad \text{on } \Gamma_2.$$

Physical parameters, exact solution, and geometries We set $k = 10$ and write everything in terms of the resolvent equation

$$\Delta U - s^2 U = 0 \qquad \text{in } \Omega_+, \qquad s = -\imath k.$$

The exact solution is an exterior radiating solution given by

$$\begin{aligned}
U(\mathbf{z}) &:= H_0^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{sc}|)\\
U_x(\mathbf{z}) &:= -H_1^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{sc}|)\frac{z_1 - x_1^{sc}}{|\mathbf{z} - \mathbf{x}^{sc}|}\\
U_y(\mathbf{z}) &:= -H_1^{(1)}(\imath s|\mathbf{z} - \mathbf{x}^{sc}|)\frac{z_2 - x_2^{sc}}{|\mathbf{z} - \mathbf{x}^{sc}|}\\
\nabla U(\mathbf{z}) &:= (U_x(\mathbf{z}), U_y(\mathbf{z})),
\end{aligned}$$

where $H_m^{(n)}(\imath s r)$ is a hankel function of the n^{th} kind of order m, and $\mathbf{x}^{sc} := (0.1, 0.2)$ is inside one of the geometries (Γ_2 in this case). The geometry is the union of an ellipse (Γ_1)

$$(5x/6)^2 + y^2 = 1$$

with a kite-shape (Γ_2), all of which fits inside the rectangle $[-1, 4.5] \times [-1.2, 1.2]$. The solution will be observed at three exterior points:

$$\mathbf{x}_1^{\text{obs}} := (1.5, 0), \quad \mathbf{x}_2^{\text{obs}} := (-1, 0), \quad \mathbf{x}_3^{\text{obs}} := (1, -1).$$

Discretization and restriction matrices We will use $N = 60$ points to discretize each geometry. The domains are sampled (three times $\varepsilon = 0, -1/6, 1/6$) and the discrete geometries are merged. The function `selectComponents` is used to produce two $1 \times N$ arrays

`g1indices`, `g2indices`

which contain the indices in the merged geometry corresponding to each respective geometry, along with two $N \times 2N$ restriction matrices

$$\mathrm{R}_1, \qquad \mathrm{R}_2$$

which, when multiplied on the right by a $2N \times 2$ array containing all the midpoints, for example, yield a $N \times 2$ array of the midpoints of only one of the geometries. The elements of the Calderón Calculus are then computed

$$\mathrm{V}_1(s), \qquad \mathrm{W}_2(s), \qquad \mathrm{Q}_2, \qquad \mathrm{K}(s), \qquad \mathrm{J}(s),$$

where $\mathrm{V}_1(s), \mathrm{W}_2(s)$ are $N \times N$ matrix operators exclusively on Γ_1, Γ_2 respectively, while $\mathrm{K}(s), \mathrm{J}(s)$ are $2N \times 2N$ matrix operators on Γ. The four operators depend on s and are output as function handles. If the field `g1.parity` exists for the first geometry then the parity matrix `g1.parity` is added to $\mathrm{V}_1(s)$, while if the field `g2.parity` exists then the absolute value of the parity matrix, |`g2.indices`| is added to $\mathrm{W}_2(s)$. We next sample the incident wave (see Chapter 3 of the deltaBEM documentation)

$$\boldsymbol{\beta}_0, \qquad \boldsymbol{\beta}_1$$

on the two boundaries. Finally we create the single and double layer potentials

$$\mathrm{S}_1(s), \qquad \mathrm{D}_2(s)$$

at the observation points, with $\mathrm{S}_1(s)$ made from the first geometry and $\mathrm{D}_2(s)$ made from the second geometry.

Integral formulation and error The approximation for the exterior trace and normal derivatives will be $\boldsymbol{\phi}_2$ and $\boldsymbol{\lambda}_1$ respectively. The effective approximation of the exterior trace (appearing in potential expressions, and also as the final approximation of the exterior trace) will be $\mathrm{Q}_2\boldsymbol{\phi}_2$. The discrete integral equations are then given in matrix form by

$$\begin{bmatrix} \mathrm{V}_1(s) & \mathrm{R}_1\mathrm{K}(s)\mathrm{R}_2^\top \\ \mathrm{R}_2\mathrm{J}(s)\mathrm{R}_1^\top & -\mathrm{W}_2(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_0 \\ \boldsymbol{\beta}_1 \end{bmatrix}.$$

We then compute the approximate potential at the observation points

$$U_{\mathrm{scat}} = \mathrm{S}_1(s)\boldsymbol{\lambda}_1 + \mathrm{D}_2(s)\mathrm{Q}_2\boldsymbol{\phi}_2,$$

and the error is computed:

$$e_{\mathrm{U}} := \max|U_{\mathrm{scat}} - U(\mathbf{x}^{\mathrm{obs}})|.$$

Plot If the user desires a plot of the solution then the value `yes = 1` is input. A triangulation $\mathcal{T}$ is produced in the box $[-2,5]\times[-1.5,1.5]$ and the single-layer and double-layer potentials are computed on the triangulation using their respective geometries, Γ_1 and Γ_2. A plot of the potential is then generated on the triangulation.