`simulationFDelasticity.m`

Frequency domain elastic scattering by three circular obstacles

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The problem. The goal of this script is to exemplify the use of deltaBEM to simulate elastic wave scattering. A direct approach, based on the second boundary integral identity, is used to solve both Dirichlet and Neumann problems. The scattered wave field $\mathbf{u}$ satisfies the equations

$$\nabla\cdot\boldsymbol{\sigma}(\mathbf{u})+k^2\rho\mathbf{u}=\mathbf{0}\qquad\text{in }\Omega_+:=\mathbb{R}^2\setminus\overline{\Omega_-}$$

with a (Dirichlet or Neumann) boundary condition

$$\gamma\mathbf{u}+\boldsymbol{\beta}_0=\mathbf{0}\qquad\text{or}\qquad\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}=\mathbf{0}\qquad\text{on }\Gamma:=\partial\Omega_+,$$

taken from a given incident displacement or stress field:

$$\boldsymbol{\beta}_0:=\gamma\mathbf{u}^{\text{inc}},\qquad\boldsymbol{\beta}_1:=\boldsymbol{\sigma}(\mathbf{u}^{\text{inc}})\mathbf{n}.$$

The solution also satisfies the Kupradze radiation conditions at infinity. Following the way of writing the equations in the deltaBEM Calderón Calculus, we will write the equation as

$$\nabla\cdot\boldsymbol{\sigma}(\mathbf{u})-s^2\rho\mathbf{u}=\mathbf{0}\qquad\text{with}\qquad s=-\imath k.$$

The domain is exterior to the disjoint union of three disks

$$\Omega_-:=B((1,1),1)\cup B((3,3),1)\cup B((3.5,0.4),1),$$

and the stress tensor is given by Hooke's law

$$\boldsymbol{\sigma}(\mathbf{u})=\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^\top)+\lambda(\nabla\cdot\mathbf{u})\,\mathbf{I}.$$

For the example we consider the physical parameters:

$$k=3\qquad s:=-\imath k,\qquad\rho=50,\qquad\lambda=5,\qquad\mu=3.$$

The incident wave is a plane pressure elastic wave

$$\mathbf{u}^{\text{inc}}(\mathbf{z}):=\exp(-\tfrac{s}{c_L}\mathbf{d}\cdot\mathbf{z}),\qquad\mathbf{d}:=(1,0),\qquad c_L:=\sqrt{\tfrac{\lambda+2\mu}{\rho}}.$$

The scattered displacement field can be written in terms of its Cauchy data

$$\mathbf{u}=\mathcal{D}(s)\boldsymbol{\phi}-\mathcal{S}(s)\boldsymbol{\lambda},\qquad\boldsymbol{\phi}:=\gamma\mathbf{u},\quad\boldsymbol{\lambda}:=\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}.$$

The boundary identity

$$\tfrac{1}{2}\boldsymbol{\lambda} = -\mathcal{W}(s)\boldsymbol{\phi} - \mathcal{J}(s)\boldsymbol{\lambda}$$

will be used as the integral equation. In the case of the Dirichlet problem, we will solve

$$\boldsymbol{\phi} = -\boldsymbol{\beta}_0, \qquad (\tfrac{1}{2} + \mathcal{J}(s))\boldsymbol{\lambda} = -\mathcal{W}(s)\boldsymbol{\phi},$$

whereas for the Neumann problem, we will solve

$$\boldsymbol{\lambda} = -\boldsymbol{\beta}_1, \qquad \mathcal{W}(s)\boldsymbol{\phi} = -(\tfrac{1}{2} + \mathcal{J}(s))\boldsymbol{\lambda}.$$

Discretization. We will use `Np` $= N = 300$ points on each of the circular scatterers. Note that the siaze of the systems will be $6N \times 6N$, since we have three obstacles and two components of the displacement or normal stress fields. The geometric sampling is carried out next. The main and the two companion grids are sampled and then merged in the lines

```
[g1,g1p,g1m]=sample(@ellipse,NP,[1 1],[1 1]);
[g2,g2p,g2m]=sample(@ellipse,NP,[1 1],[3 3]);
[g3,g3p,g3m]=sample(@ellipse,NP,[1 1],[3.5 .4]);
[g,gp,gm]=merge({g1,g2,g3},{g1p,g2p,g3p},{g1m,g2m,g3m});
```

The exterior displacement field will be evaluated on the vertices of a triangulation, inside the box $[-1, 5.5] \times [-1.5, 5]$ and outside the obstacles Ω_-. This triangulation is produced with the following lines of code

```
Box=[-1 5.5 5 -1.5];
h = 0.075;                    % Mesh diameter
[X,Y,T]=triangulateGeometry(Box,g,0.015,0.015,h,0);
T=T(:,1:3);
```

To avoid recomputing the triangulation for the same geometric configuration, whenever the physical parameters are changed, the triangulation is stored in a file. The first lines of the script use the variable `newexample` to decide whether the triangulation is produced or read from a file, and `nameoffile` to give the name (and path) of the mesh file.

The next step in the discretization consists of the sampling of the incident fields and the construction of the potentials and integral operators. The matrices Q and M for the scalar Calderón Calculus have to be used to build block diagonal counterparts $\mathbf{Q} = \mathrm{diag}(\mathrm{Q}, \mathrm{Q})$ and $\mathbf{M} = \mathrm{diag}(\mathrm{M}, \mathrm{M})$:

```
[Q,M] = CalderonCalculusMatrices(g,1);   % the fork option is activated
O = zeros(size(M));
Q = [Q O; O Q];
M = [M O; O M];
```

The operators $\mathcal{J}(s)$ and $\mathcal{W}(s)$ are next produced, and the two components of the trace (and the normal traction) of the incident field) are sampled

```
[b0x,b1x] = CalderonCalculusTest(...
                          u1,@(x1,x2) [sig11(x1,x2) sig12(x1,x2)],gp,gm,1);
[b0y,b1y] = CalderonCalculusTest(...
                          u2,@(x1,x2) [sig12(x1,x2) sig22(x1,x2)],gp,gm,1);
beta0 = [b0x;b0y];   % Trace of the displacement (for Dirichlet problem)
beta1 = [b1x;b1y];   % Normal traction (for Neumann problem)
```

Here `u1` and `u2` are the components of the incident displacement field, and `sig11`, `sig12`, `sig22` are the entries of the stress tensor corresponding to that field.

Discrete integral formulations. For the Dirichlet problem, we solve

$$\mathbf{M}\boldsymbol{\phi} = -\boldsymbol{\beta}_0, \qquad (\tfrac{1}{2}\mathbf{M} + \mathbf{J}(s))\boldsymbol{\lambda} = -\mathbf{W}(s)\boldsymbol{\phi},$$

while for the Neumann problem, we solve

$$\mathbf{M}\boldsymbol{\lambda} = -\boldsymbol{\beta}_1, \qquad \mathbf{W}(s)\boldsymbol{\phi} = -(\tfrac{1}{2}\mathbf{M} + \mathbf{J}(s))\boldsymbol{\lambda}.$$

The potential postprocessing is identical in both cases:

$$\mathbf{u} = \mathbf{D}(s)\mathbf{Q}\boldsymbol{\phi} - \mathbf{S}(s)\boldsymbol{\lambda}.$$

For the graphs, the incident displacement field is evaluated at the vertices of the triangulation and then added to the scattered field $\mathbf{u}^{\mathrm{tot}} := \mathbf{u} + \mathbf{u}^{\mathrm{inc}}$.

Some reminders. The way the Calderón Calculus is designed, there are some simple rules to be kept in mind.

- The samples of the incident wave and its normal stress on the boundary of the scatterers are considered observations and not discrete functions. They cannot, therefore, be the input of integral operators. This is the reason for the equations of the form $\boldsymbol{\phi} = -\boldsymbol{\beta}_0$ that have to be discretized with the approximation of the identity operator $\mathbf{M}\boldsymbol{\phi} = -\boldsymbol{\beta}_0$.

- Note also that the approximation of the trace $\boldsymbol{\phi}$ is not the input of the discrete double layer potential: instead, the effective density is $\mathbf{Q}\boldsymbol{\phi}$. This is common to the Calderón Calculus for all operators: quantities in $H^{1/2}$ spaces are always affected by the quadrature matrix $\mathbf{Q}$; in the case of the integral operators, the matrix is part of the integral operator.