`simulationTransmissionProblem .m`

Simulation of time-harmonic scattering by two obstacles with different material properties

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Problem. The goal of this script is to create a simulation of the scattered wave for the transmission problem

$$\Delta U + k^2 U = 0 \qquad \text{in } \Omega_+, \qquad \partial_r U - ikU = o(r^{-1/2}) \qquad \text{as } r \to \infty,$$

and

$$\Delta V_i + (\tfrac{k}{c_i})^2 V_i = 0 \qquad \text{in } \Omega_i, \quad i = 1, 2,$$

where Ω_i are the interior domains of two penetrable obstacles, with

$$\alpha_i \partial_n^- V_i = \partial_n^+ U + \gamma^+ U_{\text{inc}} \qquad \text{on } \Gamma_i, \qquad \gamma_- V_i = \gamma^+ U + \gamma^+ U_{\text{inc}}, \quad i = 1, 2$$

and

$$\beta_0 = -\gamma U^{\text{inc}}, \qquad \beta_1 = -\partial_n U^{\text{inc}}.$$

Physical parameters and geometry. The wave speeds and densities in each obstacle are chosen to be

$$\mathbf{c} = [\tfrac{1}{2} \quad 2], \qquad \boldsymbol{\alpha} = [1 \quad 1],$$

and we will write everything in terms of the resolvent equation

$$\Delta U - s^2 U = 0 \qquad s = -\imath k.$$

The incident wave is

$$\begin{aligned} U^{\text{inc}} &:= \exp(s\mathbf{d} \cdot \mathbf{z}), \\ U^{\text{inc}}_x &:= s d_x \exp(s\mathbf{d} \cdot \mathbf{z}), \\ U^{\text{inc}}_y &:= s d_y \exp(s\mathbf{d} \cdot \mathbf{z}), \end{aligned}$$

where the vector $\mathbf{d} = [d_x \quad d_y] = [\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}}]$, is the direction of the incident wave. The geometry is the union of the TV-shape (see deltaBEM documentation), which fits in a square $[-1.6, 1.6]^2$, with the ellipse

$$(x_1 - 4)^2 + \tfrac{1}{4}(x_2 - 5)^2 = 1.$$

Discretization. We will use $N_1 = N_2 = 600$ points to discretize the TV-shape and ellipse, respectively, giving $N = 1200$ points total. The domains are sampled (three times $\varepsilon = 0, -1/6, 1/6$) and the discrete geometries are merged. Next we compute the elements of the Calderón Calculus,

$$\mathrm{Q}, \quad \mathrm{M}, \quad \mathrm{V}(s), \quad \mathrm{K}(s), \quad \mathrm{J}(s), \quad \mathrm{W}(s), \quad \mathrm{V}^{\mathrm{int}}(s), \quad \mathrm{K}^{\mathrm{int}}(s), \quad \mathrm{J}^{\mathrm{int}}(s), \quad \mathrm{W}^{\mathrm{int}}(s),$$

where the internal operators are

$$\begin{bmatrix} \mathrm{W}^{\mathrm{int}}(s) & \mathrm{J}^{\mathrm{int}}(s) \\ -\mathrm{K}^{\mathrm{int}}(s) & \mathrm{V}^{\mathrm{int}}(s) \end{bmatrix} = \begin{bmatrix} \alpha_1 \mathrm{W}_1^{\mathrm{int}}(s/c_1) & 0 & \mathrm{J}_1^{\mathrm{int}}(s/c_1) & 0 \\ 0 & \alpha_2 \mathrm{W}_2^{\mathrm{int}}(s/c_2) & 0 & \mathrm{J}_2^{\mathrm{int}}(s/c_2) \\ -\mathrm{K}_1^{\mathrm{int}}(s/c_1) & 0 & \alpha_1^{-1}\mathrm{V}_1^{\mathrm{int}}(s/c_1) & 0 \\ 0 & -\mathrm{K}_2^{\mathrm{int}}(s/c_2) & 0 & \alpha_2^{-1}\mathrm{V}_2^{\mathrm{int}}(s/c_2) \end{bmatrix}$$

Here $\mathrm{V}_1, \mathrm{K}_1, \mathrm{J}_1, \mathrm{W}_1$ would be the Calderón Calculus for the first geometry. The eight operators depend on s and are output as function handles. We next sample the incident wave

$$\boldsymbol{\beta}_0, \qquad \boldsymbol{\beta}_1.$$

> From here on the script is independent of the number of obstacles. The construction of the coupled and uncoupled Calderón Calculus is also independent of the number of obstacles.

Integral formulation. The Costabel-Stephan formulation is

$$\begin{bmatrix} \mathrm{W}(s) + \mathrm{W}^{\mathrm{int}}(s) & \mathrm{J}(s) + \mathrm{J}^{\mathrm{int}}(s) \\ -\mathrm{K}(s) - \mathrm{K}^{\mathrm{int}}(s) & \mathrm{V}(s) + \mathrm{V}^{\mathrm{int}}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}^- \\ \boldsymbol{\lambda}^- \end{bmatrix} = \left[\begin{matrix} \mathrm{W}(s) & \frac{1}{2}\mathrm{M} + \mathrm{J}(s) \\ \frac{1}{2}\mathrm{M} - \mathrm{K}(s) & \mathrm{V}(s) \end{matrix} \right) \begin{bmatrix} \boldsymbol{\tau}_0 \\ \boldsymbol{\tau}_1 \end{bmatrix},$$

where

$$\mathrm{M}\boldsymbol{\tau}_0 = -\boldsymbol{\beta}_0, \qquad \mathrm{M}\boldsymbol{\tau}_1 = -\boldsymbol{\beta}_1, \qquad \boldsymbol{\phi}^+ = -\boldsymbol{\tau}_0 + \boldsymbol{\phi}^-, \qquad \boldsymbol{\lambda}^+ = -\boldsymbol{\tau}_1 + \boldsymbol{\lambda}^-, \qquad \boldsymbol{\varphi}^\pm = \mathrm{Q}\boldsymbol{\phi}^\pm,$$

and

$$\boldsymbol{\phi}^- = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_{N_{\mathrm{comp}}}]^T, \qquad \boldsymbol{\lambda}^- = [\boldsymbol{\lambda}_1^{\mathrm{prov}} \quad \boldsymbol{\lambda}_2^{\mathrm{prov}} \quad \cdots \quad \boldsymbol{\lambda}_{N_{\mathrm{comp}}}^{\mathrm{prov}}]^T,$$

with N_{comp} being the number of geometries. We then set

$$\boldsymbol{\lambda}_i = \tfrac{1}{\alpha_i}\boldsymbol{\lambda}_i^{\mathrm{prov}}.$$

The code recognizes an arbitrary number of obstacles and unpacks the $\boldsymbol{\phi}_i$ and $\boldsymbol{\lambda}_i$ as components of a cell array, where each $\boldsymbol{\phi}_i, \boldsymbol{\lambda}_i$ has N_i elements.

Triangulation and potential evaluation. If $N_{\rm comp}$ equals the number of components we generate triangulations

$$\mathcal{T}_1 \quad \text{for } \Omega_+, \qquad \mathcal{T}_{i+1} \quad \text{for } \Omega_i, \quad i = 1, ..., N_{\rm comp}.$$

The scattered wave is then computed as

$$U_h^{\rm scat} = \mathrm{D}(s)\boldsymbol{\phi}^+ - \mathrm{S}(s)\boldsymbol{\lambda}^+,$$

and the potentials inside each obstacle are

$$V_{h,i} = \mathrm{S}_i(\tfrac{s}{c_i})\boldsymbol{\lambda}_i - \mathrm{D}_i(\tfrac{s}{c_i})\boldsymbol{\phi}_i,$$

where $\mathrm{D}(s), \mathrm{S}(s)$ are the layer potentials for Γ and $\mathrm{S}_i(s), \mathrm{D}_i(s)$ are the layer potentials for Γ_i.

Plots and movies. We plot the real part of the total solution

$$U^{\rm scat} + U^{\rm inc} \quad \text{in } \Omega_+, \qquad V_i \quad \text{in } \Omega_i,$$

and save it in the `figures` folder. (Note that it is easy to plot instead the imaginary parts or the absolute values.) To create the time-harmonic movie, we plot

$$\cos(t_j)\mathrm{Re}(U^{\rm scat} + U^{\rm inc}) + \sin(t_j)\mathrm{Im}(U^{\rm scat} + U^{\rm inc}) \quad \text{on } \mathcal{T}_1,$$

and

$$\cos(t_j)\mathrm{Re}(V_i) + \sin(t_j)\mathrm{Im}(V_i) \quad \text{on } \mathcal{T}_{i+1}$$

for $t_j = \frac{2\pi}{20}j, j = 0, ..., 20$, and then save the plots in the `figures` folder. This plots the solution over one period. To increase the time to n periods with m frames per period, simply have $t_j = \frac{2\pi}{m}j$ with $j = 1, ..., mn$. An array of indices of `type:string` is created in order to label the plots.

> There are two ways of running this script. If `newexample = 1` then the triangulation is created and stored in the `meshes` folder. If `newexample = 0` then the triangulation is uploaded from the `meshes` folder. In this case, the triangulation `TPtwpScat` was uploaded.
> As the code is written, the organization of the folders is as follows: the scriptsand the meshes are placed in separate folders hanging from the same one.