First steps in the boundary element method

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Presentation

This is yet another instance of the beginning of a book I'll never find the time to finish. Instead, you'll find here some basic lectures, at an elementary level,, about boundary integral equations and boundary element methods.

Chapter 1

The single layer potential for the Laplacian

chap:1

1.1 Exterior solutions of the Laplace equation

We consider the following geometric set-up. A bounded domain $\Omega_- \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, is assumed to be on one side of its boundary Γ. Often, the domain of interest will be $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$, and will be referred to as the exterior domain. The unit normal vector field $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^d$ is well defined (almost everywhere) and assumed to point from Ω_- to Ω_+.

With respect to the boundary, the farther we will go from the point of view of regularity is Lipschitz regularity (see Section 1.5 below), but it will help the reader to place themselves in one of these extreme situations:

- Γ is a closed simple polygon in the plane or a closed polyhedron in the space that is locally representable as a graph.
- Γ is locally the graph of a $\mathcal{C}^\infty$ function.

Many interesting situations fall in-between. We will make an effort in exploring domains in their due generality, as we approach the more theoretical chapters of this book.

Most of this book is going to be focused on exterior solutions of boundary value problems. We start by focusing on the Laplace equation

$$\Delta u = 0 \qquad \text{in } \Omega_+. \tag{1.1}$$

At this point we are not going to worry about smoothness of the solution or of the boundary. Smoothness of solution will be a given, as we will see later in the book, and the boundary might have different kinds of regularity. A boundary condition will be given on Γ, typically, a Dirichlet condition

$$u = \beta_0 \qquad \text{on } \Gamma, \tag{1.2}$$

or a Neumann condition

$$\partial_\nu u := \nabla u \cdot \boldsymbol{\nu} = \beta_1 \qquad \text{on } \Gamma. \tag{1.3}$$

The problem is not complete without a certain kind of condition at infinity. At this moment, we are going to deal with the asymptotic behavior at infinity in full generality. In the two dimensional case, we assume that

$$u(\mathbf{z}) = a \log|\mathbf{z}| + b + \frac{\mathbf{c}\cdot\mathbf{z}}{|\mathbf{z}|^2} + \mathcal{O}(|\mathbf{z}|^{-2}) \qquad \text{as } |\mathbf{z}| \to \infty. \tag{1.4}$$ eq:1.4

The scalar quantities a and b, and the vector $\mathbf{c}$ might be given or not. The Landau symbol is used as follows:

$$f(\mathbf{z}) = \mathcal{O}(|\mathbf{z}|^{-m}) \text{ as } |\mathbf{z}| \to \infty$$

means that there exist $C, R > 0$ such that

$$|f(\mathbf{z})| \le C|\mathbf{z}|^{-m} \qquad \forall \mathbf{z} \quad \text{s.t. } |\mathbf{z}| \ge R.$$

In the three dimensional case, we will consider the following type of asymptotic behavior

$$u(\mathbf{z}) = \frac{a}{|\mathbf{z}|} + \frac{\mathbf{b}\cdot\mathbf{z}}{|\mathbf{z}|^3} + \mathcal{O}(|\mathbf{z}|^{-3}) \qquad \text{as } |\mathbf{z}| \to \infty. \tag{1.5}$$ eq:1.5

Asymptotic conditions at infinity, imposed to solutions of boundary value problems, are often called *radiation conditions* for reasons that will be clear when we start exploring equations related to wave propagation.

Some functional notation. Even for readers who are not fully acquainted with integration theory, it will be useful to adopt the classical notation of Lebesgue spaces. On a domain $\Omega \subset \mathbb{R}^d$ (bounded or not), we consider the spaces

$$L^1(\Omega) := \{u : \Omega \to \mathbb{R} \,:\, \int_\Omega |u| < \infty\}, \tag{1.6a}$$ eq:1.5a

$$L^2(\Omega) := \{u : \Omega \to \mathbb{R} \,:\, \int_\Omega |u|^2 < \infty\}, \tag{1.6b}$$ eq:1.5b

and

$$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} \,:\, |u| \text{ bounded}\}. \tag{1.6c}$$ eq:1.5c

Some further precision is needed. The integral in (1.6a) and (1.6b) is a Lebesgue integral. The Lebesgue measure will not be displayed unless we explicitly show the variable in the function. Functions that are equal almost everywhere (except in set of zero measure) are considered to be the same function. If the reader is not acquainted with these concepts, there are simple intuitive introductions to measure and integration in the literature. We will not attemp them here. In (1.6c), boundedness has to be understood as the existence of a constant such that $|u| \le C$ almost everywhere, i.e., except in a set of zero measure. Once again, functions that differ on sets of measure zero are considered to be the same function. When we are working on the boundary Γ, we will use a Lebesgue measure in $d-1$ dimensions mapped on the boundary (this is particularly easy to understand in the case of polygonal boundaries), in order to define the spaces $L^1(\Gamma)$ and $L^2(\Gamma)$. The space $L^\infty(\Gamma)$ is similarly defined.

1.2 The single layer potential in three dimensions

sec:1.2

Let $\eta : \Gamma \to \mathbb{R}$ be a given function defined on the boundary Γ. We then define the *single layer potential* $\mathrm{S}\eta : \mathbb{R}^3 \setminus \Gamma \to \mathbb{R}$ with the formula:

$$(\mathrm{S}\eta)(\mathbf{z}) := \int_\Gamma \frac{\eta(\mathbf{y})}{4\pi|\mathbf{z}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}).$$

In this context, the input function η will be called a *density*. Several simple properties are next listed. Their proofs are proposed in the exercise list. If $\eta \in L^1(\Gamma)$ and $u = \mathrm{S}\eta$, then $u \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \Gamma)$, and

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \Gamma, \tag{1.7a}$$

$$u(\mathbf{z}) = a\frac{1}{|\mathbf{z}|} + \mathcal{O}(|\mathbf{z}|^{-2}) \qquad \text{as } |\mathbf{z}| \to \infty. \tag{1.7b}$$

where $a = \frac{1}{4\pi}\int_\Gamma \eta \,\mathrm{d}\Gamma$. It takes quite some more work to prove the following estimates. On *smooth points* of the boundary, that is, on points $\mathbf{x} \in \Gamma$ on which there is a tangent plane, and for smooth enough density η, we have

eq:1.8

$$(\gamma^+ u)(\mathbf{x}) := \lim_{\Omega_+ \ni \mathbf{z} \to \mathbf{x}} u(\mathbf{x}) \qquad = \int_\Gamma \frac{\eta(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}), \tag{1.8a}$$

$$(\gamma^- u)(\mathbf{x}) := \lim_{\Omega_- \ni \mathbf{z} \to \mathbf{x}} u(\mathbf{x}) \qquad = \int_\Gamma \frac{\eta(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}), \tag{1.8b}$$

$$(\partial_\nu^+ u)(\mathbf{x}) := \lim_{\Omega_+ \ni \mathbf{z} \to \mathbf{x}} \nabla u(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) = -\frac{1}{2}\eta(\mathbf{x}) + \int_\Gamma \frac{(\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{1.8c}$$

$$(\partial_\nu^- u)(\mathbf{x}) := \lim_{\Omega_- \ni \mathbf{z} \to \mathbf{x}} \nabla u(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) = \frac{1}{2}\eta(\mathbf{x}) + \int_\Gamma \frac{(\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}). \tag{1.8d}$$

This shows that the single layer potential is continuous across the surface from where it is generated, but that its gradient is discontinuous in the normal direction, with a jump discontinuity equal to the density. We have used the symbols $\gamma^\pm$ to denote restrictions to the boundary from $\Omega_\pm$. These will be the precise symbols for trace operators that we will use when we get to be theoretically precise. We have also adopted the quite usual symbols $\partial_\nu^\pm$ for normal derivatives from both sides of Γ.

Smooth points on the boundary include all points that are not on edges in the case of polyhedral boundaries. All points are smooth on smooth surfaces.

Two boundary integral operators. It is quite convenient to have separate symbols for the two integral expressions in the righ-hand-sides of (1.8): for an arbitrary density η and for points $\mathbf{x} \in \Gamma$, we define

eq:1.9

$$(\mathrm{V}\eta)(\mathbf{x}) := \int_\Gamma \frac{\eta(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}), \tag{1.9a}$$

$$(\mathrm{K}^t\eta)(\mathbf{x}) := \int_\Gamma \frac{(\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}). \tag{1.9b}$$

Note that $\mathrm{S}\eta$ and $\mathrm{V}\eta$ share the same mathematical expression, although they mean two different things: $\mathrm{S}\eta$ is a function defined in free space $\mathbb{R}^3 \setminus \Gamma$ and $\mathrm{V}\eta$ is defined on the interface Γ. The operators V and K^t are our first two examples of *boundary integral operators*, respectivelly called *single layer operator* and *adjoint/transpose double layer operator*. We will need to wait until Chapter 2 to understand the name of the second one, as well as its superscripted notation.

Some jump relations. We can collect much of the previous information with the help of the jump and average operators: if $u : \mathbb{R}^d \setminus \Gamma \to \mathbb{R}$ is sufficiently smooth on both sides of Γ, we write

$$[\![\gamma u]\!] := \gamma^- u - \gamma^+ u, \qquad [\![\partial_\nu u]\!] := \partial_\nu^- u - \partial_\nu^+ u,$$

and

$$\{\!\{\gamma u\}\!\} := \tfrac{1}{2}\gamma^- u + \tfrac{1}{2}\gamma^+ u, \qquad \{\!\{\partial_\nu u\}\!\} := \tfrac{1}{2}\partial_\nu^- u + \tfrac{1}{2}\partial_\nu^+ u.$$

eq:1.10

Then, the limiting properties of the single layer potential (1.8) can be condensed as

$$[\![\gamma \mathrm{S}\eta]\!] = 0, \qquad [\![\partial_\nu \mathrm{S}\eta]\!] = \eta, \tag{1.10a}$$

$$\{\!\{\gamma \mathrm{S}\eta\}\!\} = \gamma^{\pm}\mathrm{S}\eta = \mathrm{V}\eta, \qquad \{\!\{\partial_\nu \mathrm{S}\eta\}\!\} = \mathrm{K}^t\eta. \tag{1.10b}$$

In the jargon of boundary integral equations, these are some of the *jump relations* of potentials.

A boundary integral equation. We can now try to use the single layer potential to

eq:1.11

solve the exterior Dirichlet problem:

$$\Delta u = 0 \qquad \text{in } \Omega_+, \tag{1.11a}$$

$$\gamma^+ u = \beta_0 \qquad (\text{on } \Gamma), \tag{1.11b}$$

$$u = \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty, \tag{1.11c}$$

for given boundary data $\beta_0 : \Gamma \to \mathbb{R}$, and where the simplified asymptotic notation for the radiation condition at infinity is self-explanatory. We can look then for a solution of

eq:1.12

(1.11) in the form of a single layer potential

$$u = \mathrm{S}\eta, \tag{1.12a}$$

eq:1.12a

for a density η to be determined. This function satisfies the Laplace equation and has the right asymptotic behavior at infinity. It solves the Dirichlet problem if and only if

$$\mathrm{V}\eta = \beta_0. \tag{1.12b}$$

eq:1.12b

The equation (1.12b), which has the explicit form

$$\int_\Gamma \frac{\eta(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}) = \beta_0(\mathbf{x}), \qquad \mathbf{x}\in\Gamma, \tag{1.13}$$

eq:1.13

is our first example of a *boundary integral equation.* Once again, we are not concerned with smoothness or regularity of the functions involved in this problem, so we do not have the right to talk about existence and uniqueness of solution to (1.12b). Note though, that in principle we will aim for equation (1.13) to be satisfied only almost everywhere, or, more precisely: on any point not on vertices/edges in the polygonal case, and everywhere in the case of smooth surfaces. What is important to remember at this point is that the equation (1.12b) is coupled with the integral representation (1.12a), since it is the Dirichlet problem (1.11) that is of interest to us, and not the integral equation per se. It is common to refer to a potential representation of the solution of (1.11) as a *potential ansatz.*

Single or simple? There is some discussion in the community on whether the name of the single layer potential should be simple layer potential, as it would correspond to a more direct translation of its German origin (due to Carl Friedrich Gauss), which is respected in other languages. It would also probably pair better with its counterpart, the double layer potential that we will meet in Chapter 2. (It has to be said that both simple and single convey the idea of one-ness.) On the other hand, English language usage for single layer is quite extended and unlikely to change just because we want it to change (that is probably not how languages work, especially in relatively unimportant topics as this one), so we will be faithful to the old single layer tradition.

1.3 Some numerical methods

sec:1.3

We fast forward to show two numerical approximation methods for problem (1.12b). Consider a partition of Γ in relatively open disjoint patches $\{T_1,\dots,T_N\}$

$$T_i\cap T_j=\emptyset \quad i\neq j, \qquad \Gamma=\cup_{j=1}^N \overline{T}_j,$$

and the set of piecewise constant functions with respect to this partition

$$X_h := \{\eta^h : \Gamma\to\mathbb{R} \,:\, \eta^h|_{T_j}\in\mathcal{P}_0(T_j) \quad \forall j\}, \tag{1.14}$$

where $\mathcal{P}_0(\Omega)$ is the set of constant functions on the domain Ω. (We will use a script h to denote discretization. The meaning of this index will be clear when we study approximation properties of the space X_h in different norms. In that case h will be the maximum diameter of the elements of the partition.) A basis for X_h is easy to define using the characteristic functions of the elements of the partition:

$$\chi_j := \begin{cases} 1, & \text{in } T_j,\\ 0, & \text{elsewhere.}\end{cases} \tag{1.15}$$

A Galerkin method. We look for $\eta^h \in X_h$ such that

$$\int_\Gamma \mu^h(\mathbf{x})(\mathrm{V}\eta^h)(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma \mu^h(\mathbf{x})\beta_0(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) \qquad \forall \mu^h \in X_h. \tag{1.16}$$ eq:1.16

It is clear that any solution of this equation is also a solution of

$$\int_{T_i} (\mathrm{V}\eta^h)(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_{T_i} \beta_0(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) \qquad i = 1, \ldots, N, \tag{1.17}$$ eq:1.17

since we only need to restrict testing in (1.16) to the basis functions χ_i. However, linearity of integration and the fact that $\{\chi_1, \ldots, \chi_N\}$ is a basis for X_h, shows that (1.17) also implies (1.16). We can go one step further by decomposing the unknown in the given basis. We thus look for

$$(\eta_1, \ldots, \eta_N) \in \mathbb{R}^N, \qquad \text{representing} \qquad \eta^h = \sum_{j=1}^N \eta_j \chi_j, \tag{1.18a}$$

such that

$$\sum_{j=1}^N \left(\int_{T_i} (\mathrm{V}\chi_j)(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) \right) \eta_j = \int_{T_i} \beta_0(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) \qquad i = 1, \ldots, N. \tag{1.18b}$$ eq:1.18b

It is clear from (1.18b) that we are in the presence of an $N \times N$ system of linear equations. A closer inspection to the elements of this matrix show that they are

$$V_{ij} = \int_{T_i} (\mathrm{V}\chi_j)(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_{T_i}\int_{T_j} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}). \tag{1.19}$$ eq:1.19

Therefore, the matrix in the system (1.18b) is symmetric. It is also a full matrix to the furthest extent of this concept: not a single element of the matrix is zero. It will take us a little bit longer to show that the matrix is also positive definite. This is a consequence of a coercivity estimate that we will not see for the time being. Once the discrete density η^h has been computed (by assembling the system (1.18b) and solving it afterwards), the discrete solution to the Dirichlet problem (1.11) is given by the single layer potential representation

$$u^h := \mathrm{S}\eta^h := \sum_{j=1}^N \left(\int_{T_j} \frac{1}{4\pi|\cdot - \mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}) \right) \eta_j.$$

This is a linear combination of N single layer potentials generated from constant density distributions on each of the patches T_j. One interesting feature of this kind of methods is the fact that u^h still satisfies the differential equation, and it is only the boundary condition that has been approximated. More precisely, since $u^h = \mathrm{S}\eta^h$, then

$$\Delta u^h = 0 \qquad \text{in } \mathbb{R}^3 \setminus \Gamma, \tag{1.20a}$$

$$u^h = \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty. \tag{1.20b}$$

The boundary condition is not satisfied. In its place we have the following bits of information on the boundary:

$$[\![\gamma u^h]\!] = 0, \qquad [\![\partial_\nu u^h]\!] \in X_h, \qquad \int_\Gamma \mu^h(\gamma^+ u^h - \beta_0)\mathrm{d}\Gamma = 0 \quad \forall \mu^h \in X_h. \tag{1.20c}$$

The first of these conditions stems from the fact that single layer potentials are continuous across their boundary. The second one just restates the fact that the jump of the normal derivative, that is, the density, was found in the subspace X_h. The third condition repeats the fact that we are forcing some moments of $\gamma^+ u^h = \mathrm{V}\eta^h$ and β_0 to coincide, namely, those moments obtained by averaging on the elements T_i.

A comment on notation. The matrix (1.19) is symmetric and therefore the order used to write integration domains and variables in that formula is not very relevant. However, from now on we will keep the following easy to remember convention: assuming that the order of integration is not relevant, we will write

$$\int_{T_i}\int_{T_j} \Phi(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) = \int_{T_i}\left(\int_{T_j} \Phi(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})\right)\mathrm{d}\Gamma(\mathbf{x}),$$

that is, we will write the integration variables in the same order as the domains.

Computation of integrals. At the present stage of this text, we are going to ignore the difficulties of computing the integrals

$$\int_{T_i}\int_{T_j} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) \qquad \text{and} \qquad \int_{T_i} \beta_0(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}).$$

Even in the case of a polyhedral surface (where element integrals become integrals over plane domains quite easily), partitioned into simple shapes T_i (quadrilaterals or triangles), this computation is not an easy task. Assume for the moment being that Γ is a polyhedron and that it has been subdivided into triangles in the tradition of Finite Element triangulations: two adjacent triangles can share a common vertex or a common full edge. We have four situations for pairs of elements (T_i, T_j):

- $T_i = T_j$,
- $\overline{T}_i$ and $\overline{T}_j$ share a common edge,
- $\overline{T}_i$ and $\overline{T}_j$ share a common vertex,
- $\overline{T}_i$ and $\overline{T}_j$ are disjoint.

Only the latter case is easy to handle from the point of view of the integral in (1.19). In this case the integrand is a very smooth function of both variables and we can think of easy ways to approximate the double integral. (This might be misleadingly easy, however, when the elements are disjoint, but they are geometrically very close, since the function gets to have very large values and gradients.) All other cases involve integrating functions with singularities. More about this in the appendices.

The collocation method. Let us go back to our general partition $\{\Gamma_1, \ldots, \Gamma_N\}$ and to the space X_h. We now choose points $\mathbf{x}_i \in T_i$ for all i (one point on each element), and look for $\eta^h \in X_h$ satisfying

$$(\mathrm{V}\eta^h)(\mathbf{x}_i) = \beta_0(\mathbf{x}_i), \qquad i = 1, \ldots, N. \tag{1.21}$$

Equivalently, we can look for coefficients $(\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$ satisfying

$$\sum_{j=1}^{N} \left(\int_{T_j} \frac{1}{4\pi|\mathbf{x}_i - \mathbf{y}|} \mathrm{d}\Gamma(\mathbf{y}) \right) \eta_j = \beta_0(\mathbf{x}_i) \qquad i = 1, \ldots, N. \tag{1.22}$$

We are again in the presence of an $N \times N$ linear system. We have lost symmetry in the discretization process, but have greatly simplified several aspects: the right-hand-side does not require any integration process, and the matrix involves only integration over T_j, as opposed to integration in double the number of variables that appear in Galerkin methods. It has to be said though, that for this particular equation, even restricting to simpler boundaries and partitions, there is no satisfactory theory supporting the collocation method, while the theory of Galerkin methods is very well understood since over four decades ago, including results for superconvergence that we will mention in due time.

1.4 The two dimensional case

sec:1.4

While the same ideas that we have explained in Section 1.2 will work in two dimensional problems, and being able to foresee that the corresponding integral equations will be simpler (integration on Γ is essentially one-dimensional), the unbounded behavior of the fundamental solution of the Laplace equation in two dimensions complicates asymptotics at infinity and the search for well-posed boundary value problems. We start with the definition of the single layer potential

$$(\mathrm{S}\eta)(\mathbf{z}) = -\frac{1}{2\pi} \int_\Gamma \log|\mathbf{z} - \mathbf{y}| \eta(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}). \tag{1.23}$$

The difference between this potential and its three dimensional counterpart is the function

$$-\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{y}| \qquad \text{instead of} \qquad \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

This function is the *fundamental solution*, or Green's function in free space, for the Laplacian (properly speaking, for the minus Laplacian). For very similar reasons that were argued in Section 1.2 (details are requested as exercises), if $u = \mathrm{S}\eta$ with $\eta \in L^1(\Gamma)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \Gamma)$ and

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Gamma, \tag{1.24a}$$

$$u = a \log r + \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty, \tag{1.24b}$$

where $a = -\frac{1}{2\pi}\int_\Gamma \eta \, d\Gamma$. The limits of $S\eta$ and its normal derivative on both sides of Γ can be represented using two integral operators:

eq:1.25

$$(V\eta)(\mathbf{x}) := -\frac{1}{\pi}\int_\Gamma \log|\mathbf{x}-\mathbf{y}|\eta(\mathbf{y})d\Gamma(\mathbf{y}), \tag{1.25a}$$

$$(K^t\eta)(\mathbf{x}) := \int_\Gamma \frac{(\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})}{2\pi|\mathbf{x}-\mathbf{y}|^2}\eta(\mathbf{y})d\Gamma(\mathbf{y}). \tag{1.25b}$$

Like in the three dimensional case, when we think of the integral operators, we have to remember that the input is a function defined on Γ and the output is a function defined on Γ too. The jump relations are exactly the same as in the three dimensional case, namely

$$[\![\gamma S\eta]\!] = 0, \qquad [\![\partial_\nu S\eta]\!] = \eta, \tag{1.26a}$$

eq:1.26a

and

$$\{\!\{\gamma S\eta\}\!\} = \gamma^{\pm}S\eta = V\eta, \qquad \{\!\{\partial_\nu K^t\eta\}\!\} = K^t\eta. \tag{1.26b}$$

eq:1.26b

Equalities are to be understood as happening only on smooth points of the boundary and for smooth enough densities. From the rightmost formulas in (1.26a) and (1.26b), it is easy to prove that

$$\partial_\nu^+ S\eta = -\tfrac{1}{2}\eta + K^t\eta, \qquad \partial_\nu^- S\eta = \tfrac{1}{2}\eta + K^t\eta.$$

eq:1.27

An integral equation for the Dirichlet problem. Consider the exterior problem

$$\Delta u = 0 \qquad \text{in } \Omega_+, \tag{1.27a}$$

$$\gamma^+ u = \beta_0 \qquad (\text{on } \Gamma), \tag{1.27b}$$

$$u = u_\infty + \mathcal{O}(r^{-1}) \qquad \text{as } r\to\infty, \tag{1.27c}$$

where $u_\infty \in \mathbb{R}$ is unknown. This is a good model for the exterior Dirichlet problem, looking for bounded solutions. It will be seen in future chapters that the option of taking $u = \mathcal{O}(r^{-1})$ as the radiation condition leads to problems that might not have a solution. Based on what we know about the single layer potential, we propose the following representation for the solution of (1.27)

eq:1.28

$$u = S\eta + u_\infty, \qquad \text{where} \quad \int_\Gamma \eta \, d\Gamma = 0. \tag{1.28a}$$

This guarantees that u is a solution of the exterior Laplace equation with the correct asymptotic behavior at infinity. Both η and u_∞ are unknown. The boundary condition is then equivalent to

$$V\eta + u_\infty = \beta_0. \tag{1.28b}$$

A Galerkin method. Let now $\{T_1, \ldots, T_N\}$ be a non-overlapping partition of Γ in the same conditions as in Section 1.3 and let X_h be equally defined. We then look for $\eta^h \in X_h$ and $u^h_\infty \in \mathbb{R}$ such that

eq:1.29

$$\int_\Gamma \mu^h \left(\mathrm{V}\eta^h + u^h_\infty\right)\mathrm{d}\Gamma = \int_\Gamma \mu^h \beta_0 \,\mathrm{d}\Gamma \qquad \forall \mu^h \in X_h \tag{1.29a}$$

and

$$\int_\Gamma \eta^h \,\mathrm{d}\Gamma = 0. \tag{1.29b}$$

Using the basis of characteristic functions on the elements of the partition $\{\chi_1, \ldots, \chi_N\}$, we can look for $(\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$ and $u^h_\infty \in \mathbb{R}$ such that

eq:1.30

$$\sum_{j=1}^N V_{ij}\eta_j + |T_i| u^h_\infty = \int_{T_i} \beta_0(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) \qquad i = 1, \ldots, N \tag{1.30a}$$

$$\sum_{j=1}^N |T_j|\eta_j \qquad\qquad = 0 \tag{1.30b}$$

where

$$V_{ij} = V_{ji} = -\frac{1}{2\pi}\int_{T_i}\int_{T_j} \log|\mathbf{x}-\mathbf{y}|\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) \tag{1.30c}$$

and $|T_i|$ is the length of the element T_i. The global matrix is symmetric but has a not very attractive zero in the last diagonal term. An exercise below suggests a simple way around this. As for implementation of this problem, we propose some approximations of the integrals in one of this chapter's projects.

The interior problem. One interesting feature of this single layer representation of the solution of the Dirichlet problem is that if (η, u_∞) solve (1.28), the interior part of the potential representation (after all $\mathrm{S}\eta + u_\infty$ is well defined in $\mathbb{R}^2 \setminus \Gamma$) is also a solution of

$$\Delta u = 0 \quad \text{in } \Omega_-, \qquad \gamma^- u = \beta_0.$$

1.5 Lipschitz domains (*)

sec:1.A.1

In this section we are going to describe with precision what the type of domains are, for which we will be delivering precise mathematical statements about the boundary element method. Where and why some of the hypotheses of the definition are used is not easy to grasp unless you are willing to go deep in the theory of Sobolev spaces. The goal of this course is not that, but it is necessary to have delimited the scope of our theory and to warn of the dangers of more complicated situations.

Let then Ω_- be a bounded open domain in $\mathbb{R}^d$ with boundary Γ and exterior Ω_+. Assume now that for every point $\mathbf{x} \in \Gamma$ (and here every means, exactly for every point), we can find:

(a) a vector $\mathbf{a} \in \mathbb{R}^d$ and an orthogonal matrix Q, determining a rigid motion

$$\Phi(\mathbf{x}) = Q\mathbf{x} + \mathbf{a},$$

(b) a Lipschitz function $h : \mathbb{R}^{d-1} \to \mathbb{R}$, that is, a function for which there exists L such that

$$|h(\widetilde{\mathbf{x}}) - h(\widetilde{\mathbf{y}})| \leq C|\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}| \qquad \forall \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \in \mathbb{R}^{d-1},$$

(c) and a cylindrical reference domain

$$\{(\widetilde{\mathbf{x}}, t) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, |\widetilde{\mathbf{x}}| \leq \varepsilon, |t| \leq \rho\},$$

so that

$$\begin{aligned} \Phi(\widetilde{\mathbf{x}}, h(\widetilde{\mathbf{x}}) + t) \in \Omega_+, \qquad & |\widetilde{\mathbf{x}}| \leq \varepsilon, \qquad 0 < t \leq \rho, \\ \Phi(\widetilde{\mathbf{x}}, h(\widetilde{\mathbf{x}})) \in \Gamma, \qquad & |\widetilde{\mathbf{x}}| \leq \varepsilon, \\ \Phi(\widetilde{\mathbf{x}}, h(\widetilde{\mathbf{x}}) - t) \in \Omega_+, \qquad & |\widetilde{\mathbf{x}}| \leq \varepsilon, \qquad 0 < t \leq \rho. \end{aligned}$$

(The elements of the above list –rigid motion, Lipschitz graph, and localization cylinder– depend on the point $\mathbf{x}$.) In this case we say that Ω_- is a (strongly) Lipschitz domain. We also say that Γ is locally a Lipschitz graph and Ω_- is locally a Lipschitz hypograph (it can be locally placed under a Lipschitz graph.) Note that the local process to observe/describe a Lipschitz hypograph consists of the following steps (see Figure 1.1):

- create a (possibly small) localization cylinder;
- deform the cylinder in the vertical x_d direction using the Lipschitz function;
- apply a rigid motion to place this locally deformed cylinder in the physical space, making the image of the original horizontal disk $(\widetilde{\mathbf{x}}, 0)$ part of the boundary, the negative part being mapped to the interior domain, and the positive part to the exterior domain.

In some theoretical expositions it is common to flip the x_d axis so that the domain is a hypergraph. This obviously does not change the kind of domains.

Examples. Most friendly domains are Lipschitz domains. Circles, rectangles, simple polygons,... are planar Lipschitz domains. Domains with internal or boundary cracks are not Lipschitz domains. Domains with cuspidal points on the boundary are not Lipschitz domains. Similarly, spheres, torii, hexahedra,... are Lipschitz domains in the space, and once again, cracked domains are excluded. (This does not mean that they cannot be dealt with at all, but that much of what we are going to say here has to be adapted, sometimes in highly nontrivial ways, to these more general situations.)

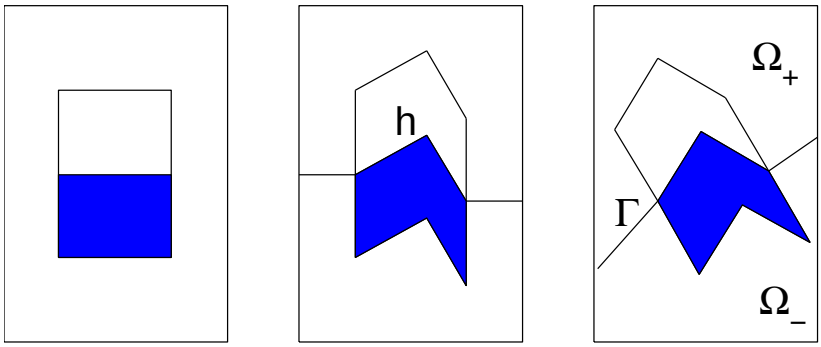

Figure 1.1: The localization process in the definition of a Lipschitz domain. The cylinder on the left. Its deformation by a part of a Lipschitz graph in the center. The deformed cylinder being rigidly mapped to physical coordinates. fig:1.1

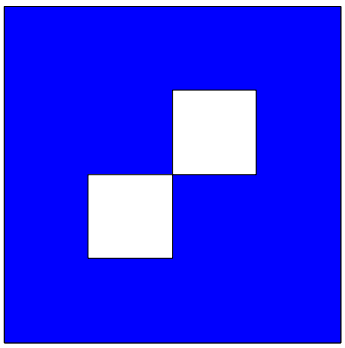

Figure 1.2: A non-Lipschitz planar polygon. The central point of the figure is troublesome, since the domain cannot be placed on one side of the boundary at this point.

The normal vector. Due to a well-known but really deep theorem by Rademacher, the function h is differentiable almost everywhere in its domain of definition. This means that we can attach a tangent plane to almost every point in the graph $\Phi(\widetilde{\mathbf{x}}, h(\widetilde{\mathbf{x}}))$. The unit normal vector at the point $\mathbf{z} = (\widetilde{\mathbf{x}}, h(\widetilde{\mathbf{x}}))$, pointing towards Ω_+ will be denoted $\boldsymbol{\nu}(\mathbf{z})$. It is not complicated to see that the normal vector at the point exists and coincides even if we change the point $\mathbf{x}$ around which we are localizing. We thus get an almost everywhere defined unit vector field $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^d$. In the case of a polyhedral domain, the construction of the normal vector field is completely straightforward.

Smooth domains. If we still assume the same construction as in the definition of Lipschitz domain, but increase the smoothness of the function h in step (b), then we get smoother versions of the definition. In particular, if instead of a Lipschitz function h,

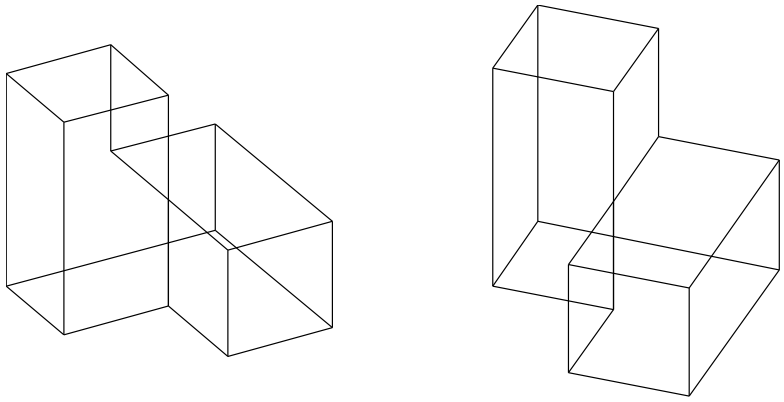

Figure 1.3: Two views of a popular non-Lipschitz polyhedron, made up of to corssing bricks. The crossing point is the only point where we cannot take a point of view allowing us to locally view the boundary as a graph. Note that 'topologically' the figure is quite simple, but this lack of visibility of the boundary is enough for the domain not to fit in the hypotheses. The domain is however an example of weakly Lipschitz domain.

we take h to be an infinitely often differentiable function, we obtain a smooth domain. Several interesting geometric constructions can be carried out for smooth domains (or for sufficiently smooth domains, meaning with h several times continuously differentiable). The normal vector $\boldsymbol{\nu}(\mathbf{x})$ is well defined for every point $\mathbf{x} \in \Gamma$. We can then find $\varepsilon > 0$ such that the set of points

$$\{\mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x}) \,:\, \mathbf{x} \in \Gamma, \quad |t| \leq \varepsilon\}$$

satisfy $\mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x}) = \mathbf{y} + s\,\boldsymbol{\nu}(\mathbf{y})$ if and only if $\mathbf{x} = \mathbf{y}$ and $s = t$. In this tubular domain, we can extend the normal vector field to a function $\boldsymbol{\nu}(\mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x})) = \boldsymbol{\nu}(\mathbf{x})$ (for $\mathbf{x} \in \Gamma$, $|t| \leq \varepsilon$), that is, the normal vector field can be constantly extended among the normal direction. This extension is a smooth function in its domain of definition.

1.6 Literature, exercises and working projects

1. (Section 1) Let $u : \Omega_+ \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth up to the boundary and satisfying (1.4). Give conditions on a, b and $\mathbf{c}$ for u to be in $L^2(\Omega_+)$.

2. (Section 1) Let $u : \Omega_+ \subset \mathbb{R}^3 \to \mathbb{R}$ be smooth up to the boundary and satisfying (1.4). Give conditions on a and $\mathbf{b}$ for u to be in $L^2(\Omega_+)$.

3. (Section 2 – Needs analysis) Show that if $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^\infty$ in the set

$\{(\mathbf{z},\mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, \mathbf{z} \neq \mathbf{y}\}$, and $\eta \in L^1(\Gamma)$, then

$$u(\mathbf{z}) := \int_\Gamma \Phi(\mathbf{z},\mathbf{y})\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})$$

is in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \Gamma)$ and can be arbitrarily often differentiated under integral sign. Use this to show that the single layer potential in two and three dimensions is a smooth function and that it satisfies the Laplace equation.

4. (Section 2) Prove that

$$\left|\frac{1}{|\mathbf{z}-\mathbf{y}|} - \frac{1}{|\mathbf{z}|}\right| \leq \frac{C_R}{|\mathbf{z}|^2}, \qquad |\mathbf{z}| \geq 2R \geq 2|\mathbf{y}|.$$

Use it to prove that the three dimensional single layer potential satisfies

$$(\mathrm{S}\eta)(\mathbf{z}) = \frac{1}{4\pi|\mathbf{z}|}\int_\Gamma \eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) + \mathcal{O}(|\mathbf{z}|^{-2}) \qquad \text{as } |\mathbf{z}| \to \infty.$$

5. (Section 2) The previous exercise gives an explicit formula for the first asymptotic term at infinity of the single layer potential. Find an explicit expression of the second one, that is, the term you have to subtract to the single layer potential to get an $\mathcal{O}(|\mathbf{z}|^{-3})$ remainder.

6. (Section 2) Let

$$E_d(\mathbf{x},\mathbf{y}) := \begin{cases} -\dfrac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|, & (d=2), \\ \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, & (d=3). \end{cases}$$

Show that

$$\nabla_{\mathbf{y}} E_d(\mathbf{x},\mathbf{y}) = \frac{1}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}(\mathbf{x}-\mathbf{y}).$$

7. (Section 4) Show that

$$\frac{C_{1,R}}{|\mathbf{x}|} \leq \log|\mathbf{x}-\mathbf{y}| - \log|\mathbf{x}| \leq \frac{C_{2,R}}{|\mathbf{x}|}, \qquad |\mathbf{x}| \geq 4R \geq 4|\mathbf{y}|.$$

Use it to prove that in two dimensions

$$(\mathrm{S}\eta)(\mathbf{z}) = -\frac{1}{2\pi}\log|\mathbf{z}|\int_\Gamma \eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) + \mathcal{O}(|\mathbf{z}|^{-1}) \qquad \text{as } |\mathbf{z}| \to \infty.$$

8. (Section 4) **Exact solutions of the Laplace equation.** Let $\mathbf{x}_1, \mathbf{x}_2$ be two distinct points in Ω_-. Show that

$$u(\mathbf{x}) = \log\frac{|\mathbf{x}-\mathbf{x}_1|}{|\mathbf{x}-\mathbf{x}_2|}$$

is an exterior decaying solution of the Laplace equation in two dimensions. Show that $1, x_1, x_2, x_1^2 - x_2^2, x_1x_2$ are also solutions to the interior Laplace equation. Find more polynomial solutions in two and three dimensions. (Polynomials satisfying the Laplace equation are called harmonic polynomials.)

9. (Section 4) Instead of the given basis for X_h, we can use the following modified basis

$$\widetilde{\chi}_i := \tfrac{1}{|T_i|}\chi_i - \tfrac{1}{|\Gamma_{i+1}|}\chi_{i+1}, \quad i = 1, \ldots, N-1, \qquad \widetilde{\chi}_N := \chi_1 + \ldots + \chi_N = 1.$$

If we represent the solution of (1.29) as

$$\eta^h = \sum_{j=1}^{N} \widetilde{\eta}_j \widetilde{\chi}_j,$$

show that the system can be decoupled into the computation of the coefficients $\widetilde{\eta}_j$ followed by the computation of u_∞^h. Write down the corresponding matrix. Finally, relate the coefficients $\{\widetilde{\eta}_j\}$ to the coefficients $\{\eta_j\}$ of the decomposition with respect to the original basis of X_h.

10. (Section 4) Assume that we have been able to solve

$$\mathrm{V}\eta + u_\infty = \beta_0, \qquad \int_\Gamma \eta \, \mathrm{d}\Gamma = c,$$

for given β_0 and $c \in \mathbb{R}$. What problem is $u = \mathrm{S}\eta + u_\infty$ solving in this case? (Note that in this case, the leading asymptotic behavior is unbounded, but given.)

11. (Section 4) **The logarithmic capacity.** In this exercise we are going to assume that the problem

$$\mathrm{V}\phi + a = 0, \qquad \int_\Gamma \phi \, \mathrm{d}\Gamma = 1,$$

admits a unique solution. We then define $C_\Gamma := \exp(-a)$. The quantity a is called Robin's constant for Γ and C_Γ is called the logarithmic capacity of Γ. The function ϕ is called the equilibrium distribution for Γ.

   (a) Show that C_Γ is invariant by translations, plane symmetries and rotations of Γ. (Hint. Write the system that defines (ϕ, a) explicitly and apply translations, symmetries and rotations to Γ.)

   (b) If $c\Gamma := \{c\mathbf{x} : \mathbf{x} \in \Gamma\}$ with $c > 0$, show that $C_{c\Gamma} = c\, C_\Gamma$, which means that the logarithmic capacity scales like a diameter.

   (c) For the given solution (ϕ, a), we define $u = \mathrm{S}\phi$. Show that u is constant in Ω_-. (This requires a uniqueness argument for the solution of the interior Laplacian that you can assume.)

   (d) Show that if $C_\Gamma = 1$, then V is not injective.

   (e) Show that the logarithmic capacity of a disk is its radius. (This one is quite difficult. It is easy to use a symmetry argument to guess that ϕ has to be constant. Using this and an exact computation, it is possible to find the solution for the case of the circle of radius one.)

Project # 1.1 – A quadrature method (coding)

Statement of the problem. We are looking for a numerical solution for the interior/exterior Dirichlet problem for the Laplacian

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \qquad \gamma^{\pm} u = \beta_0, \qquad u = u_\infty + \mathcal{O}(r^{-1}) \quad \text{at infinity,}$$

in the case when Γ is a smooth parametrizable curve in the plane. A boundary integral formulation for this problem has been given in Section 1.4. and the exercise list contains several exact solutions for both interior and exterior problems. Our starting point is a 1-periodic parametrization of the boundary $\mathbf{x} : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$, with the following properties

$$\mathbf{x}(t+1) = \mathbf{x}(t) \quad \forall t, \qquad |\mathbf{x}'(t)| \neq 0 \quad \forall t, \qquad \mathbf{x}(t) \neq \mathbf{x}(\tau) \quad t - \tau \notin \mathbb{Z}.$$

We next parametrize equations (1.28). We abuse notation by renaming

$$\beta_0(t) := \beta_0(\mathbf{x}(t)), \qquad \eta(t) := \eta(\mathbf{x}(t))|\mathbf{x}'(t)|.$$

eq:project1 The integral system (1.28) is then equivalent to the periodic integral equation

$$-\frac{1}{2\pi}\int_0^1 \log|\mathbf{x}(t) - \mathbf{x}(\tau)|\,\eta(\tau)\mathrm{d}\tau + u_\infty = \beta_0(t) \qquad \forall t \tag{1.31a}$$

eq:project1a

with the side condition

$$\int_0^1 \eta(\tau)\mathrm{d}\tau = 0 \tag{1.31b}$$

and the potential representation

$$u(\mathbf{z}) = -\frac{1}{2\pi}\int_0^1 \log|\mathbf{z} - \mathbf{x}(\tau)|\,\eta(\tau)\,\mathrm{d}\tau + u_\infty. \tag{1.31c}$$

A quadrature method. The method to approximate (1.31) is simple. We approximate every integral by the trapezoidal rule on a uniform grid, and collocate (1.31a) in pairs of strategically chosen points. We thus choose an integer N, define $h := 1/N$ and consider the parametric points

$$t_j := j\,h, \qquad t_i^+ := (i + \tfrac{1}{6})\,h, \qquad t_i^- := (i - \tfrac{1}{6})\,h. \tag{1.32}$$

eq:project1.2

For ease of notation we will write $\sum_{\pm} a_{\pm} = \frac{1}{2}(a_+ + a_-)$ for the average of quantities tagged with the $\pm$ signs. The unknowns are then $\eta_j \approx h\,\eta(t_j)$ and the system is:

project1.3

$$-\frac{1}{2\pi}\sum_{\pm}\sum_{j=1}^{N}\log|\mathbf{x}(t_i^{\pm}) - \mathbf{x}(t_j)|\eta_j + u_\infty^N = \sum_{\pm}\beta_0(t_i^{\pm}), \qquad i = 1,\ldots,N, \tag{1.33a}$$

$$\sum_{j=1}^{N}\eta_j = 0. \tag{1.33b}$$

The solution is postprocessed to a discrete potential

$$u^N(\mathbf{z}) = -\frac{1}{2\pi}\sum_{j=1}^{N} \log|\mathbf{z} - \mathbf{x}(t_j)|\,\eta_j + u^N_\infty. \tag{1.33c}$$

Estimate the errors

$$|u(\mathbf{z}) - u^N(\mathbf{z})| = \mathcal{O}(N^{-3})$$

for different points $\mathbf{z}$ and increasing values of N. Try with different geometries and different points. Show that as $\mathbf{z}$ gets closer to the boundary the errors degenerate. The $\pm 1/6$ lateral displacement in the observation points $t_i^{\pm}$ in (1.32) can be changed to be a parameter $\pm\varepsilon$. Try other choices and compare orders of convergence. For a given value of N graph the error as a function of the parameter ε.

Another way of getting to the same method. We can devise a Galerkin method for (1.31) based on the space

$$X_h := \{\eta : \mathbb{R} \to \mathbb{R} \,:\, \eta(1+\cdot) = \eta, \quad \eta|_{(t_i - h/2, t_i + h/2)} \in \mathcal{P}_0 \quad \forall i\}.$$

Show that the discrete equations can be understood as a full discretization of the Galerkin equations using the midpoint quadrature for one of the integrals and a two point quadrature for the other one. (Note that the use of different quadrature rules makes the matrix non-symmetric.)

Project # 1.2 – A Galerkin procedure (coding)

Statement of the problem. Let Γ be a simple closed polygon in the plane. We are looking for a numerical solution for the interior/exterior Dirichlet problem for the Laplacian

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \qquad \gamma^{\pm} u = \beta_0, \qquad u = u_\infty + \mathcal{O}(r^{-1}) \quad \text{at infinity,}$$

using the Galerkin equations (1.30) leading to a potential representation

$$u^h(\mathbf{z}) = \sum_{j=1}^{N} \Phi_j(\mathbf{z})\eta_j + u^h_\infty, \qquad \Phi_j(\mathbf{z}) := -\frac{1}{2\pi}\int_{T_j} \log|\mathbf{z}-\mathbf{y}|\mathrm{d}\Gamma(\mathbf{y}).$$

An exact computation. Let e be a segment joining points $\mathbf{v}$ and $\mathbf{w}$, and let $\mathbf{z} \in \mathbb{R}^2$. Then

$$\int_e \log|\mathbf{z}-\mathbf{y}|\mathrm{d}\Gamma(\mathbf{y}) = f(\beta,\eta) - f(\alpha,\eta), \tag{1.34}$$

eq:project2.1

where

$$f(t,\eta) := \frac{t}{2}\log(t^2+\eta^2) + \eta \arctan\left(\frac{t}{\eta}\right) - t,$$

and

$$\begin{aligned} \ell &:= |\mathbf{w}-\mathbf{v}|, & \eta &:= (\mathbf{z}-\mathbf{v})\cdot\mathbf{n}, \\ \mathbf{t} &:= \frac{1}{\ell}(\mathbf{w}-\mathbf{v}), & \alpha &:= -(\mathbf{z}-\mathbf{v})\cdot\mathbf{t}, \\ \mathbf{n} &:= \mathbf{t}^{\perp} = (t_2,-t_1), & \beta &:= \alpha+\ell = -(\mathbf{z}-\mathbf{w})\cdot\mathbf{t}. \end{aligned}$$

Prove (1.34).

Gaussian quadrature. Let e be again a segment joining the points $\mathbf{v}$ and $\mathbf{w}$. Then we can approximate

$$\int_e \rho(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \approx \frac{|\ell|}{2}(\rho(\mathbf{g}^+) + \rho(\mathbf{g}^-)), \tag{1.35}$$

eq:project2.2

where

$$\mathbf{g}^{\pm} = \tfrac{1}{2}(\mathbf{v}+\mathbf{w}) \pm \tfrac{1}{2\sqrt{3}}(\mathbf{w}-\mathbf{v}).$$

The method. The Galerkin equations (1.30) need approximation of some integrals. For the right-hand-side, use Gaussian integration as in (1.35). For the matrix, compute the following approximation

$$V_{ij} \approx V_{ij}^{\mathrm{ns}} := -\frac{1}{2\pi}\frac{|T_i|}{2}\left(\int_{T_j} \log|\mathbf{g}_i^+ - \mathbf{y}|\mathrm{d}\Gamma(\mathbf{y}) + \int_{T_j} \log|\mathbf{g}_i^- - \mathbf{y}|\mathrm{d}\Gamma(\mathbf{y})\right),$$

where $\mathbf{g}_i^{\pm}$ are the Gaussian points in the element T_i, and the integrals can be computed analytically using (1.34). Finally symmetrize $V_{ij} \approx \frac{1}{2}(V_{ij}^{\mathrm{ns}} + V_{ji}^{\mathrm{ns}})$.

Book-keeping. One of the most complicated parts of numerical coding is having the right data structures. The partition of the boundary can be described with two matrices.

- The first one is an $N \times 2$ matrix with the coordinates of all nodes of the partition of Γ. We will refer to it as coord. Then $\mathrm{coord}(i,:) = [x, y]$ are the coordinates of the node number i. The matrix itself imposes the ordering of the nodes.
- The second one is a matrix with the element references. It is equally an $N \times 2$ matrix with positive integers as entries: if $\mathrm{ele}(i,:) = [2,\ 5]$, this says that the element T_i starts at the point $\mathbf{v}_2$ and ends at the point $\mathbf{v}_5$. It is convenient that numbering of elements is done with positive orientation, that is, if you go from the first vertex to the second, the domain lies to the left-hand-side of the element.

As a first part of your code you will have to create some code to generate partitions of a given polygon, generating the coordinate and the element matrices.

Experiments. Fix a polygon and create a sequence of partitions (uniform is fine in a first approach). The computations needed for obtain $u^h(\mathbf{z})$ can be carried out with analytic integration as in (1.34). Compare then

$$|u^h(\mathbf{z}) - u(\mathbf{z})|.$$

There are several exact solutions for interior and exterior problems in the exercise list.

Project # 1.3 – The Helmholtz equation

The single layer potential. Consider the following function

$$\Phi(\mathbf{z}) := \frac{e^{\imath\omega|\mathbf{z}|}}{4\pi|\mathbf{z}|}.$$

Show that the function $u(\mathbf{z}) := \Phi(\mathbf{z} - \mathbf{x}_0)$ satisfies

$$\Delta u + \omega^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{\mathbf{x}_0\}.$$

This equation is called the *Helmholtz equation* and the function u above is called the *outgoing fundamental solution* of the Helmholtz equation. Show that u satisfies the following radiation condition at infinity

$$\partial_r u(\mathbf{z}) - \imath\omega u(\mathbf{z}) = o(|\mathbf{z}|^{-1}) \qquad \text{as } |\mathbf{z}| \to \infty,$$

where the radial partial derivative is defined as

$$\partial_r u(\mathbf{z}) = \nabla u(\mathbf{z}) \cdot \left(\tfrac{1}{|\mathbf{z}|}\mathbf{z}\right),$$

and the little o Landau symbol has to understood in the following way: $f(\mathbf{z}) = o(|\mathbf{z}|^{-m})$, when $|\mathbf{z}|^m |f(\mathbf{z})| \to 0$ as $|\mathbf{z}| \to \infty$, uniformly in all directions. This condition at infinity is called the Sommerfeld radiation condition. The single layer potential for the Helmholtz equation is defined as

$$(\mathrm{S}\eta)(\mathbf{z}) := \int_\Gamma \frac{e^{\imath\omega|\mathbf{z}-\mathbf{y}|}}{4\pi|\mathbf{z}-\mathbf{y}|}\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}).$$

Show that $\mathrm{S}\eta$ defines a smooth solution of $\Delta u + \omega^2 u = 0$ in $\mathbb{R}^3 \setminus \Gamma$, satisfying the Sommerfeld radiation condition at infinity.

A more challenging question. Study the behavior of the difference of the Helmholtz and Laplace potentials

$$w(\mathbf{z}) := \int_\Gamma \frac{e^{\imath\omega|\mathbf{z}-\mathbf{y}|} - 1}{4\pi|\mathbf{z}-\mathbf{y}|}\eta(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}).$$

In particular, if $\mathbf{x}_0 \in \Gamma$ is a smooth point on the boundary, study the limiting behavior of w and ∇w as $\mathbf{z} \to \mathbf{x}_0$ from both sides of the boundary.

Relation to the wave equation. Assume that U is a solution of the wave equation

$$\partial_t^2 U - \Delta U = 0$$

that can be written in the form

$$U(\mathbf{z}, t) = \mathrm{Re}\left(e^{-\imath\omega t} u(\mathbf{z})\right) = \cos(\omega t)\mathrm{Re}\, u(\mathbf{z}) + \sin(\omega\, t)\mathrm{Im}\, u(\mathbf{z}).$$

Show that u is a solution of the Helmholtz equation. If $u = \Phi(\cdot - \mathbf{x}_0)$, show that the corresponding function U is an spherical kind of wave moving away from $\mathbf{x}_0$. If we consider instead $u = \overline{\Phi(\cdot - \mathbf{x}_0)}$, study what the radiation condition at infinity is, and how the time-domain function U behaves.

The two dimensional case. Repeat as many of the previous arguments as you can with the two dimensional Helmholtz equation. The fundamental solution uses a Hankel function of the first kind and order zero:

$$\frac{\imath}{4} H_0^{(1)}(\omega|\mathbf{z}|),$$

and the associated radiation condition is

$$\partial_r u(\mathbf{z}) - \imath\omega u(\mathbf{z}) = o(|\mathbf{z}|^{-1/2}) \qquad \text{as } |\mathbf{z}| \to \infty.$$

This part of the project will require you to inquire on basic properties of the Hankel functions (their derivatives, their behavior at infinity, etc).

Chapter 2

Green's representation formula

chap:2

2.1 The double layer potential

Let us start by reminding ourselves about the fundamental solution for the Laplacian

$$E_d(\mathbf{x},\mathbf{y}) = \begin{cases} -\dfrac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|, & (d=2),\\ \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, & (d=3), \end{cases}$$

and about a simple computation that was previously proposed as an exercise:

$$\nabla_{\mathbf{y}} E_d(\mathbf{x},\mathbf{y}) = \frac{1}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}(\mathbf{x}-\mathbf{y}).$$

For a given density $\psi : \Gamma \to \mathbb{R}$, the *double layer potential* is defined by the formula

$$(\mathrm{D}\psi)(\mathbf{z}) := \int_\Gamma \nabla_{\mathbf{y}} E_d(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})\,\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma \frac{(\mathbf{z}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{z}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}).$$

In the same way that the single layer potential is a continuous distribution of monopoles on the boundary Γ, the double layer potential can be understood as a continuous distributrion of dipoles oriented in the normal direction. The idea of the dipole is simple: it is the limit of the scaled difference of potentials generated by two opposing charges

$$\nabla_{\mathbf{y}} E_d(\mathbf{x},\mathbf{y})\cdot\mathbf{d} = \lim_{h\to 0}\frac{1}{2h}\Big(E_d(\mathbf{x},\mathbf{y}+h\mathbf{d}) - E_d(\mathbf{x},\mathbf{y}-h\mathbf{d})\Big).$$

This argument can be stretched to let us understand the double layer potential as the limit of two single layer potentials with the same density of parallel surfaces as the distance of the surfaces decreases. As usual, some properties are easy: if $\psi \in L^1(\Gamma)$ and we define $u = \mathrm{D}\psi$, then $u \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \Gamma)$, and

eq:2.2

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{2.1a}$$

$$u = \mathcal{O}(r^{-d+1}) \qquad \text{as } r \to \infty. \tag{2.1b}$$

eq:2.3

The side limits of the double layer potential are given by the following (more complicated to prove) relations, valid on smooth points $\mathbf{x} \in \Gamma$ and for smooth enough densities:

$$(\gamma^+ \mathrm{D}\psi)(\mathbf{x}) = \frac{1}{2}\psi(\mathbf{x}) + \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.2a}$$

$$(\gamma^- \mathrm{D}\psi)(\mathbf{x}) = -\frac{1}{2}\psi(\mathbf{x}) + \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}). \tag{2.2b}$$

We will not give a closed formula for the side normal derivatives, but be content with saying at this moment that

$$\partial_\nu^+ \mathrm{D}\psi = \partial_\nu^- \mathrm{D}\psi. \tag{2.2c}$$

Two more boundary operators. We thus define two associated boundary integral operators

$$(\mathrm{K}\psi)(\mathbf{x}) := \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.3a}$$

$$(\mathrm{W}\psi)(\mathbf{x}) := -\partial_{\boldsymbol{\nu}(\mathbf{x})} \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}). \tag{2.3b}$$

While it would be extremely tempting to move the normal derivative in the definition of W inside the integral, the result would be a function with a very strong singularity that is not integrable any more. The kind of non-integrable singularity that would thus be obtained justifies naming this operator the *hypersingular operator*. There is a possible way of writing the operator using finite part integrals. We will not follow that route here, and will circumvent the problem using other possible expressions. The operator K is called the *double layer operator*. The negative sign in the definition of W might look somewhat bizarre, but is very convenient from the point of view of positivity of the operator, as we will see in due time. Note that, at least formally,

$$\begin{aligned}\int_\Gamma (\mathrm{K}\psi)(\mathbf{x})\eta(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) &= \int_\Gamma\int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\eta(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) \\ &= \int_\Gamma \psi(\mathbf{y})(\mathrm{K}^t\eta)(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}),\end{aligned}$$

which justifies the notation for K^t and its being called the adjoint double layer operator. Once again, and because it is important, remember that potentials take input (densities) on the boundary and build functions in the entire space (in $\mathbb{R}^d\backslash\Gamma$ more properly speaking), while the output of the integral operators is a function defined on the boundary.

The jump relations. Equations (2.1) and definitions (2.2) can be written together in the form of the jump relations of the double layer potential:

$$[\![\gamma \mathrm{D}\psi]\!] = -\psi, \qquad [\![\partial_\nu \mathrm{D}\psi]\!] = 0, \tag{2.4a}$$

$$\{\!\{\gamma \mathrm{D}\psi\}\!\} = \mathrm{K}\psi, \qquad \{\!\{\partial_\nu \mathrm{D}\psi\}\!\} = \partial_\nu^\pm \mathrm{D}\psi = -\mathrm{W}\psi. \tag{2.4b}$$

Is there anything we can do with the minus signs? There are two details in the limiting and jump relations of potentials that have annoyed authors since time immemorial. The first one is the prevalence of the $\frac{1}{2}$ factors in the limiting relations

$$\partial_\nu^\pm \mathrm{S}\eta = \mp\tfrac{1}{2}\eta + \mathrm{K}^t\eta, \qquad \gamma^\pm \mathrm{D}\psi = \pm\tfrac{1}{2}\psi + \mathrm{K}\psi.$$

This is easily avoided by changing the definition of all the potentials and operators. Instead of working with $E_d(\mathbf{x},\mathbf{y})$, we just need to work with $2E_d(\mathbf{x},\mathbf{y})$ everywhere. Of course, what changes then are the jump relations, since jumps of the potentials yield twice the densities. This is actually done by many mathematically inclined authors, but it is far from being the standard. The reader should be warned that every time they approach any writing with potentials, the first thing to be done is to be sure that the definitions are the same. (The letters chosen for potentials and operators also vary from author to author.) The second incovenience comes from the minus sign in the definition of W and the fact that $[\![\gamma \mathrm{D}\psi]\!] = -\psi$. Every now and then, someone comes with the idea of changing the sign in the definition of the double layer potential. The minus sign then reappears in the definition of K (or can be moved to the definition of K^t, since it would not be reasonable to call transpose to the minus transpose of an operator). That would be a minor inconvenience. Another one would occur in the next page, when the last of Green's identities suddenly stops being a commutator formula. The veredict is unclear, and uses are unlikely to change, so we will stick to more traditional definitions.

An energy free solution. The interior Neumann problem for the Laplace equation has a one dimensional kernel, namely the solutions of

$$\Delta u = 0 \quad \text{in } \Omega_-, \qquad \partial_\nu^- u = 0$$

are constant functions. In the way we will impose behavior at infinity for the exterior Neumann problem, constants will not be a problem. This effect passes onto the double layer potential. Consider the function

$$u(\mathbf{z}) = (\mathrm{D1})(\mathbf{z}) = \int_\Gamma \nabla_{\mathbf{y}} E_d(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}).$$

If $\mathbf{z} \in \Omega_+$, then we can apply the divergence theorem to the smooth vector field $\mathbf{v}(\mathbf{y}) := \nabla_{\mathbf{y}} E_d(\mathbf{z},\mathbf{y})$ to obtain

$$\begin{aligned}(\mathrm{D1})(\mathbf{z}) &= \int_\Gamma \mathbf{v}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \\ &= \int_{\Omega_-} \operatorname{div} \mathbf{v} = \int_{\Omega_-} \Delta E_d(\mathbf{z},\cdot) = 0.\end{aligned}$$

By looking at the exterior limits, we obtain

$$\tfrac{1}{2} + \mathrm{K1} = \gamma^+ \mathrm{D1} \quad = 0, \tag{2.5a}$$

$$\mathrm{W1} = -\partial_\nu^+ \mathrm{D1} = 0. \tag{2.5b}$$

eq:2.5a

Let us now turn our attention to what happens in Ω_-. The interior part of the potential $u := (\mathrm{D}1)|_{\Omega_-}$ satisfies $\Delta u = 0$ in Ω_- (it is a potential after all), and also $\partial_\nu^- u = \partial_\nu^+ \mathrm{D}1 = 0$. Therefore u is constant in Ω_-. But we even know more:

$$\gamma^- u = \gamma^- \mathrm{D}1 = -\tfrac{1}{2} + \mathrm{K}1 = -1$$

by (2.5a). All of this proves that

$$\mathrm{D}1 = -\chi_{\Omega_-} = \begin{cases} -1, & \text{in } \Omega_-, \\ 0 & \text{in } \Omega_+. \end{cases}$$

2.2 Green's Third Identity

Some background. Let us agree that everyone with some calculus background can easily remember the divergence theorem of Gauss:

$$\int_{\Omega_-} \operatorname{div} \mathbf{p} = \int_\Gamma \mathbf{p} \cdot \boldsymbol{\nu} \mathrm{d}\Gamma. \tag{2.6}$$ eq:2.6

The difficulties for this theorem are on how smooth $\mathbf{p}$ and Ω_- have to be for this equation to hold. When we get to Sobolev spaces we will see how a clean cut definition seems to solve the problem. (As usual the devil will be in the details.) If we apply this equality to a vector field of the form $v\,\mathbf{q}$ and use Leibnitz's rule to compute the divergence of the product, we get

$$\int_{\Omega_-} (\nabla v \cdot \mathbf{q} + v \operatorname{div} \mathbf{q}) = \int_\Gamma v\, \mathbf{q} \cdot \boldsymbol{\nu} \mathrm{d}\Gamma. \tag{2.7}$$ eq:2.7

In the Partial Differential Equation community (especially among numericians), this formula is often referred to as *integration by parts*. The reason is simple: in a one dimensional domain $\Omega_- = (a, b)$, the divergence theorem (2.6) is Barrow's rule (the connection between integration and differentiation)

$$\int_a^b f' = f(b) - f(a)$$

(the minus sign is due to the normal vector), and (2.7) is

$$\int_a^b f'g + fg' = f(b)g(b) - f(a)g(a),$$

which written in the slightly different way

$$\int_a^b f'g = f\,g\Big|_a^b - \int_a^b fg' +$$

becomes the popular integration by parts formula. We can now take $\mathbf{q} = \nabla u$ in (2.7) to obtain

$$\int_{\Omega_-} (\nabla u \cdot \nabla v + v\,\Delta u) = \int_\Gamma v \partial_\nu u \,\mathrm{d}\Gamma, \tag{2.8}$$ eq:2.8

which is *Green's First Identity.* This formula is the key in the weak understanding of the Laplacian that is needed to justify much of what is happening in these preliminary chapters. It is also the bread and butter of Finite Element theorists and practitioners. If we apply (2.8) to the pairs (u, v) and (v, u) and subtract the result, we obtain *Green's Second Identity*

$$\int_{\Omega_-} (v\Delta u - u\Delta\, v) = \int_\Gamma (v\partial_\nu u - u\partial_\nu v)\mathrm{d}\Gamma. \tag{2.9}$$ eq:2.9

An intuitive presentation of the Third Identity. Choose now u satisfying $\Delta u = 0$ and $v = E_d(\mathbf{z}, \cdot)$ with $\mathbf{z} \in \Omega_+$. Then (2.9) tells us that

$$\int_\Gamma E_d(\mathbf{z}, \mathbf{y})\partial_\nu u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) - \int_\Gamma \nabla_{\mathbf{y}} E_d(\mathbf{z}, \mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y}) u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = 0, \qquad \text{when } \mathbf{z} \in \Omega_+.$$

We can read this formula using potentials: it says that

$$\mathrm{S}\partial_\nu^- u - \mathrm{D}\gamma^- u = 0 \qquad \text{in } \Omega_+.$$

If $\mathbf{z} \in \Omega_-$, the argument is fuzzier. We will admit the following fact (that we will eventually prove):

$$-\int_{\Omega_-} u(\mathbf{y})\, \Delta_{\mathbf{y}} E_d(\mathbf{z}, \mathbf{y})\mathrm{d}\mathbf{y} = u(\mathbf{z}). \tag{2.10}$$ eq:2.10

This is often written using Dirac delta distributions as

$$-\Delta E_d(\mathbf{z}, \cdot) = \delta_{\mathbf{z}}.$$

Plugging (2.10) in Green's Second Identity (2.9), we obtain

$$\int_\Gamma E_d(\mathbf{z}, \mathbf{y})\partial_\nu u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) - \int_\Gamma \nabla_{\mathbf{y}} E_d(\mathbf{z}, \mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y}) u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = u(\mathbf{z}), \qquad \text{when } \mathbf{z} \in \Omega_-.$$

So far, what we have shown (this is not a proof though) is that if $\Delta u = 0$ in Ω_-, then

$$\mathrm{S}\partial_\nu^- u - \mathrm{D}\gamma^- u = \begin{cases} u, & \text{in } \Omega_-, \\ 0, & \text{in } \Omega_+. \end{cases} \tag{2.11}$$ eq:2.11

To figure out what happens on the boundary, we take the jump relations and compute starting in (2.11)

$$\begin{aligned} \gamma^- u &= \gamma^- \mathrm{S}\partial_\nu^- u - \gamma^- \mathrm{D}\gamma^- u \\ &= \mathrm{V}\partial_\nu^- u - (\tfrac{1}{2}\gamma^- u + \mathrm{K}\gamma^- u) \end{aligned}$$

which can be rearranged to yield

$$\mathrm{V}\partial_\nu^- u - \mathrm{K}\gamma^- u = \tfrac{1}{2}\gamma^- u. \tag{2.12}$$ eq:2.12

The collection of (2.11) and (2.12) is what is known as *Green's Third Identity.* It is often presented as follows:

$$\int_\Gamma E_d(\mathbf{z}, \mathbf{y})\partial_\nu u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) - \int_\Gamma \nabla_{\mathbf{y}} E_d(\mathbf{z}, \mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y}) u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \begin{cases} u(\mathbf{z}), & \mathbf{z} \in \Omega_-, \\ \frac{1}{2}u(\mathbf{z}), & \mathbf{z} \in \Gamma, \\ 0, & \mathbf{z} \in \Omega_+, \end{cases} \tag{2.13}$$ eq:2.13

with the serious warning that the equality in points of the boundary holds only on smooth points (those with a tangent plane) and that we have assumed that $\Delta u = 0$ in Ω_-. We could have also started with a more general u so that the term Δu stays in the formula, that is, in the right-hand-side of (2.13) we have to add the term

$$\int_{\Omega_-} E_d(\mathbf{z}, \mathbf{y}) \Delta u(\mathbf{y}) \mathrm{d}\mathbf{y}.$$

A different formulation. Since we care about exterior problems, our next step would be trying to justify (2.13) for exterior domains. There are two things to take into account: (a) the normal vector points inwards now, so this will force a change of signs in the formula; (b) at the very beginning we are using the divergence theorem to move to the boundary, but we have to pay attention to behavior at infinity to be sure that the integration by parts process is meaningful. Instead, we are going to take a different approach by stating (not proving or arguing) a Green's Third Identity in space. (There will be some keys to readjust in the two dimensional case.) Let then u satisfy

$$\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma, \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity.} \tag{2.14}$$

eq:2.15

If u is smooth enough (conditions will have to wait to Chapter **??**), then

$$u = \mathrm{S}[\![\partial_\nu u]\!] - \mathrm{D}[\![\gamma u]\!]. \tag{2.15a}$$

eq:2.15a

Using the jump relations of potentials, it follows that on the boundary

$$\{\!\{\gamma u\}\!\} = \mathrm{V}[\![\partial_\nu u]\!] - \mathrm{K}[\![\gamma u]\!], \tag{2.15b}$$
$$\{\!\{\partial_\nu u\}\!\} = \mathrm{K}^t[\![\partial_\nu u]\!] + \mathrm{W}[\![\gamma u]\!]. \tag{2.15c}$$

Green's Third Identity is the collection of all equations (2.15) or just the representation formula (2.15a), which implies the other two identities.

Green's Third Identity for exterior domains. Consider now u such that

$$\Delta u = 0 \quad \text{in } \Omega_+, \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity,}$$

where we are again in the three dimensional case. Let us extend u by zero to the interior domain, while keeping the same name. Then $[\![\gamma u]\!] = -\gamma^+ u$, $[\![\partial_\nu u]\!] = -\partial_\nu^+ u$, and the representation formula (2.15a) says now

$$-\mathrm{S}\partial_\nu u^+ + \mathrm{D}\gamma^+ u = \begin{cases} u, & \text{in } \Omega_+, \\ 0, & \text{in } \Omega_-. \end{cases} \tag{2.16a}$$

On the boundary, we now have the identities

$$\tfrac{1}{2}\gamma u^+ = -\mathrm{V}\partial_\nu^+ u + \mathrm{K}\gamma^+ u, \tag{2.16b}$$

eq:2.16b

$$\tfrac{1}{2}\partial_\nu^+ u = -\mathrm{K}^t \partial_\nu^+ u - \mathrm{W}\gamma^+ u. \tag{2.16c}$$

A two dimensional version of Green's Third Identity. Assume now that

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \qquad u = a + \mathcal{O}(r^{-1}) \quad \text{at infinity.}$$

Then

$$\int_\Gamma \partial_\nu^+ u \mathrm{d}\Gamma = 0. \tag{2.17}$$

Note that in any dimension

$$\int_\Gamma \partial_\nu^- u \mathrm{d}\Gamma = \int_{\Omega_-} \Delta u = 0,$$

by the divergence theorem. Therefore, now we have

$$\int_\Gamma [\![\partial_\nu u]\!] \mathrm{d}\Gamma = 0.$$

The representation formula is now

$$u = \mathrm{S}[\![\partial_\nu u]\!] - \mathrm{D}[\![\gamma u]\!] + a,$$

and the corresponding integral identities follow from the jump relations

$$\begin{aligned} \{\!\{\gamma u\}\!\} &= \mathrm{V}[\![\partial_\nu u]\!] - \mathrm{K}[\![\gamma u]\!] + a, \\ \{\!\{\partial_\nu u\}\!\} &= \mathrm{K}^t[\![\partial_\nu u]\!] + \mathrm{W}[\![\gamma u]\!]. \end{aligned}$$

All the other arguments for the three dimensional case still apply.

2.3 A direct boundary integral equation

sec:2.3

A direct method. Let us go back to the exterior Dirichlet problem in three dimensions:

eq:2.17

$$\Delta u = 0 \qquad \text{in } \Omega_+, \tag{2.18a}$$
$$\gamma^+ u = \beta_0 \qquad (\text{on } \Gamma), \tag{2.18b}$$
$$u = \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty. \tag{2.18c}$$

In Section 1.2 we explored a single layer potential based formulation

$$u = \mathrm{S}\eta, \qquad \text{where} \qquad \mathrm{V}\eta = \beta_0.$$

In this formulation η is just an unknown density, with not much relation to the problem at hand, apart from being able to deliver the solution via an integral representation. This kind of boundary integral formulation, where we propose an integral representation and derive an associated boundary integral equation is called an *indirect method*. Instead we can do the following. First let us name the *Cauchy data*

eq:2.18

$$\varphi := \gamma^+ u, \qquad \lambda := \partial_\nu^+ u. \tag{2.19a}$$

Next, we use Green's identity as the potential representation:

$$u = \mathrm{D}\varphi - \mathrm{S}\lambda. \tag{2.19b}$$

Finally, we look at the exterior trace of this latter representation and impose the known boundary value:

$$\tfrac{1}{2}\varphi = \mathrm{K}\varphi - \mathrm{V}\lambda, \qquad \varphi = \beta_0. \tag{2.19c}$$

eq:2.18c

Equation (2.19c) can be reorganized to look

$$\mathrm{V}\lambda = -\tfrac{1}{2}\varphi + \mathrm{K}\varphi = (-\tfrac{1}{2}\mathrm{I} + \mathrm{K})\varphi, \qquad \varphi = \beta_0, \tag{2.20}$$

eq:2.19

where we have now introduced an identity operator I. We are going to keep this slightly illogical notation of keeping two names for what is the same quantity $\varphi = \beta_0$. This will allow us to later unify some formulas but, even more importantly, we are going to consider the possibility of discretizing data, so that φ will end up being an approximation to β_0. There are two novelties in (2.20). First of all, the data (or a copy/approximation of the data) appear under the action of an integral operator in the right-hand-side of the equation. This is obviously more work for us to code. Secondly, λ is now a physical variable ($\lambda = \partial_\nu^+ u$), which might have its own interest, independently of the integral representation that yields the solution of the exterior Dirichlet problem. A formulation based on Green's formula, using partial Cauchy data as unknowns, is called a *direct method*.

Discretization. In principle, we could address the discretization of (2.20) in exactly the same form that we used in Section 1.3. We start with a partition of Γ in elements $\{\Gamma_1, \ldots, \Gamma_N\}$, consider the space X_h of piecewise constant functions and discretize, either with a Galerkin method

$$\lambda^h \in X_h, \qquad \int_\Gamma \mu^h (\mathrm{V}\lambda^h)\mathrm{d}\Gamma = -\frac{1}{2}\int_\Gamma \mu^h \beta_0 \mathrm{d}\Gamma + \int_\Gamma \mu^h (\mathrm{K}\beta_0)\mathrm{d}\Gamma \qquad \forall \mu^h \in X_h,$$

or with a collocation method

$$\lambda^h \in X_h, \qquad (\mathrm{V}\lambda^h)(\mathbf{x}_i) = -\frac{1}{2}\beta_0(\mathbf{x}_i) + (\mathrm{K}\beta_0)(\mathbf{x}_i) \qquad i = 1, \ldots, N.$$

In both cases the matrix is the same as in the indirect formulation. What changes is the right-hand-side, that now incorporates the need to evaluate an integral operator.

Further discretization. It is common in some communities to address the discretization of all the integral operators as if they were acting on discrete quantities, and then project or interpolate data onto the discrete spaces. We are going to explore this idea briefly now. To explain better what we want to do, we need to restrict to a more particular case. We assume Γ to be a polyhedron that has been partitioned into N triangles $\{T_1, \ldots, T_N\}$. We can then consider the usual space X_h and the space

$$Y_h := \{\psi^h \in \mathcal{C}(\Gamma) \,:\, \psi^h|_{T_j} \in \mathcal{P}_1(T_j) \quad \forall j\}.$$

Since the elements T_j are flat pieces of a surface, the meaning of the space $\mathcal{P}_1(T_j)$ is not complicated: it is just the space of polynomials of two variables and degree up to one, written in tangential coordinates. In other words, if T_j is a triangle with vertices $\mathbf{v}_\ell^j$ for $\ell = \{1, 2, 3\}$, then

$$\boldsymbol{\phi}_j(s,t) = \mathbf{v}_1^j + s(\mathbf{v}_2^j - \mathbf{v}_1^j) + t(\mathbf{v}_3^j - \mathbf{v}_1^j)$$

can be chosen as a parametrization of T_j from the reference element

$$\widehat{T} = \{(s,t) \,:\, s,t \geq 0, s+t \leq 1\}.$$

Then

$$\mathcal{P}_1(T_j) = \{p : T_j \to \mathbb{R} \,:\, p \circ \boldsymbol{\phi}_j \in \mathcal{P}_1(\widehat{\Gamma})\}.$$

This process of sending a space from a reference element to each physical element is called *pushing forward*. A basis for Y_h is easy to construct. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_M\}$ be a numbering of all vertices of the triangulation. To each vertex $\mathbf{v}_j$ we can associated a function $\psi_j \in Y_h$ satisfying

$$\psi_j(\mathbf{v}_i) = \delta_{ij}.$$

This is a simple basis of Y_h. We can then discretize (2.20) in the following way. We first construct

$$\varphi^h \in Y_h \qquad \varphi^h(\mathbf{v}_i) = \beta_0(\mathbf{v}_i) \qquad i = 1, \ldots, M, \tag{2.21a}$$

which gives the explicit representation

$$\varphi^h = \sum_{j=1}^{M} \beta_0(\mathbf{v}_j)\psi_j.$$

We then look for $\lambda^h \in X_h$ satisfying

$$\lambda^h \in X_h, \qquad \int_\Gamma \mu^h(\mathrm{V}\lambda^h)\mathrm{d}\Gamma = -\frac{1}{2}\int_\Gamma \mu^h\varphi^h\mathrm{d}\Gamma + \int_\Gamma \mu^h(\mathrm{K}\varphi^h)\mathrm{d}\Gamma \qquad \forall \mu^h \in X_h. \tag{2.21b}$$

There are three matrices involved in this discretization process. The first one is the Galerkin matrix (1.19) associated to the operator V, namely

$$V_{ij} = \int_\Gamma \chi_i(\mathrm{V}\chi_j)\mathrm{d}\Gamma = \int_{T_i}\int_{T_j} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}).$$

The second matrix is rectangular $N \times M$ with elements

$$M_{ij} = \int_\Gamma \chi_i\psi_j\mathrm{d}\Gamma = \int_{T_i}\psi_j = \begin{cases} \frac{1}{3}|T_i| & \text{if } T_i \subset \Xi_j, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Xi_j = \cup\{\Gamma_\ell : \mathbf{v}_j \in \overline{\Gamma}_\ell\}$$

is the patch of triangles sharing $\mathbf{v}_j$ as a vertex. The third matrix is

$$K_{ij} = \int_\Gamma \chi_i(\mathrm{K}\psi_j)\mathrm{d}\Gamma = \int_{T_i}\int_{\Xi_j} \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^2}\psi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}).$$

The global system is then

$$\sum_{j=1}^{N} V_{ij}\lambda_j = \sum_{j=1}^{M}(-\tfrac{1}{2}M_{ij} + K_{ij})\beta_0(\mathbf{v}_j).$$

The matrix M_{ij} is very sparse, containing a very reduced number of elements in each row. The matrix K_{ij} is full once again, although not as full as the matrix V_{ij}, in this case if T_i and Ξ_j are contained in the same face of Γ, the element K_{ij} vanishes.

2.4 An equation of the second kind

For reasons that we will explore in the exercise list, in order to use a double layer potential representation of the solution of the Dirichlet problem, it is convenient to think first of the interior problem:

eq:2.22F

$$\Delta u = 0 \qquad \text{in } \Omega_-, \tag{2.22a}$$

$$\gamma^- u = \beta_0 \qquad (\text{on } \Gamma). \tag{2.22b}$$

eq:2.22

If we represent the solution with a double layer potential

$$u = \mathrm{D}\psi, \tag{2.23a}$$

for a density to be determined, the problem is equivalent to the integral equation

$$-\tfrac{1}{2}\psi + \mathrm{K}\psi = \beta_0. \tag{2.23b}$$

eq:2.22b

Equation (2.23b) is an example of an *integral equation of the second kind.* An integral equation of the second kind is an equation of the form $f + \mathrm{F}f = g$, where F is an integral operator. (It is clear that we can modify (2.23b) to have exacly this form.) Note that once functional analysis is thrown on the collection of integral equations that we will be dealing with, some equations will be considered of the second kind only in a formal sense, depending on the properties of the integral operator F. The concept of equation of the second kind is opposed to the concept of equation of the first kind, which is any integral equation of the form $\mathrm{F}f = g$. We have seen this kind of equation before.

The origin of it all. The problem of existence of solution to the Dirichlet problem for the Laplace equation was one that bugged mathematicians for many decades. This was way before the arrival of weak derivatives, distributions, Sobolev spaces, and a wide array of modern mathematical techniques that redefined the question before answering it. Gauss had the idea of looking for solutions in the form of a double layer potential. With that he was ensuring that the Laplace equation was being solved, but then existence of solution was transferred to existence of solution to the integral equation (2.23b). This ended up being quite a difficult problem for general domains. The theory of integral equations of the second kind sparked new definitions (Hilbert space, compact operator) and general theorems (the Fredholm alternative). We will inevitably meet many of these concepts as we proceed in this course.

The double layer operator. Let us write equation (2.23b) explicitly

$$-\frac{1}{2}\psi(\mathbf{x}) + \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2^{d-1}\pi|\mathbf{x}-\mathbf{y}|^d}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \beta_0(\mathbf{x}), \qquad \mathbf{x}\in\Gamma. \tag{2.24}$$ eq:2.24

The equation is only imposed on smooth points of the boundary. Consider now the function

$$D(\mathbf{x},\mathbf{y}) := \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}.$$

If Γ is smooth enough, this function is continuous, and even smooth. Note that (2.24) can be written as

$$-\frac{1}{2}\psi(\mathbf{x}) + \frac{1}{2\pi}\int_\Gamma D(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \beta_0(\mathbf{x}) \qquad \forall \mathbf{x}\in\Gamma$$

in the two dimensional case, and

$$-\frac{1}{2}\psi(\mathbf{x}) + \int_\Gamma \frac{D(\mathbf{x},\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \beta_0(\mathbf{x}) \qquad \forall \mathbf{x}\in\Gamma$$

in the three dimensional case. We will do some additional work for the first of the two equations. It can be seen that the integral operator in the three dimensional case looks like a smooth non-symmetric variant of the single layer operator.

A Nyström method. Let us focus then in (2.23) for the case of a smooth two dimensional boundary. We equip ourselves with a quadrature formula on Γ:

$$\int_\Gamma \phi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \approx \sum_{j=1}^N \omega_j\phi(\mathbf{y}_j). \tag{2.25}$$ eq:2.25

We then approximate the integral equation (2.23b) using the quadrature formula to approximate the integral sign:

$$-\frac{1}{2}\psi^h(\mathbf{x}) + \sum_{j=1}^N \frac{\omega_j}{2\pi}\frac{(\mathbf{x}-\mathbf{y}_j)\cdot\boldsymbol{\nu}(\mathbf{y}_j)}{|\mathbf{x}-\mathbf{y}_j|^2}\psi^h(\mathbf{y}_j) = \beta_0(\mathbf{x}) \qquad \forall \mathbf{x}\in\Gamma. \tag{2.26}$$ eq:2.26

We then collocate (evaluate) (2.26) on the points $\mathbf{x}=\mathbf{y}_i$. If we rename $\psi_j := \psi^h(\mathbf{y}_j)$, we then show that any solution of (2.26) solves

$$-\frac{1}{2}\psi_i + \sum_{j=1}^N \frac{\omega_j}{2\pi}\frac{(\mathbf{y}_i-\mathbf{y}_j)\cdot\boldsymbol{\nu}(\mathbf{y}_j)}{|\mathbf{y}_i-\mathbf{y}_j|^2}\psi_j = \beta_0(\mathbf{y}_i). \tag{2.27}$$ eq:2.27

This is an $N\times N$ system of linear equations that will, we hope, determine the values ψ_j. Once we have them, we can revert (2.26) to define

$$\psi^h(\mathbf{x}) = -2\beta_0(\mathbf{x}) + \sum_{j=1}^N \frac{\omega_j}{\pi}\frac{(\mathbf{x}-\mathbf{y}_j)\cdot\boldsymbol{\nu}(\mathbf{y}_j)}{|\mathbf{x}-\mathbf{y}_j|^2}\psi_j \qquad \forall \mathbf{x}\in\Gamma. \tag{2.28}$$ eq:2.28

This is a reconstruction formula that defines $\psi^h : \Gamma \to \mathbb{R}$ from its point values. In summary, if ψ^h satisfies (2.26), then its point values satisfy (2.27). Reciprocally, if the values $(\psi_1, \ldots, \psi_N)$ satisfy (2.27) and we define ψ^h with (2.28), then $\psi^h(\mathbf{y}_j) = \psi_j$ and ψ^h satisfies (2.26). This is called a *Nyström or quadrature method*. We remark that diagonal elements of the matrix in (2.27) need to be evaluated using a limit argument. We will show explicit formulas in one project below, for the particular case of parametrizable curves.

Recovering the potential. Once we have computed ψ^h we have two options for the potential. We can use $u^h = \mathrm{D}\psi^h$, that is

$$\begin{aligned} u^h(\mathbf{z}) &= -2\int_\Gamma \frac{(\mathbf{z}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2\pi|\mathbf{z}-\mathbf{y}|^2}\beta_0(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ &\quad + \sum_{j=1}^N \frac{\omega_j}{\pi}\psi_j \int_\Gamma \frac{(\mathbf{z}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2\pi|\mathbf{z}-\mathbf{y}|^2}\frac{(\mathbf{y}-\mathbf{y}_j)\cdot\boldsymbol{\nu}(\mathbf{y}_j)}{|\mathbf{y}-\mathbf{y}_j|^2}\mathrm{d}\Gamma(\mathbf{y}) \end{aligned}$$

or we can take advantage of the quadrature formula again (2.25) to propose another approximation

$$u^h(\mathbf{z}) = \sum_{j=1}^N \omega_j \frac{(\mathbf{z}-\mathbf{y}_j)\cdot\boldsymbol{\nu}(\mathbf{y}_j)}{2\pi|\mathbf{z}-\mathbf{y}_j|^2}\,\psi_j.$$

2.5 Literature, exercises, and working projects

1. (Section 1 – Needs analysis) Show that for any $\psi \in L^1(\Gamma)$, $\mathrm{D}\psi \in \mathcal{C}^\infty(\mathbb{R}^d\setminus\Gamma)$ and that $\Delta u = 0$ in $\mathbb{R}^d \setminus \Gamma$. (Hint. You can apply Exercise 1.3 to the functions $\partial_{y_j} E_d(\mathbf{x},\mathbf{y})$ with densities $\eta\,\nu_j$, where $\boldsymbol{\nu} = (\nu_1, ..., \nu_d)$.)

2. (Section 1) The fundamental solution for the Helmholtz operator $u \mapsto \Delta u + \omega^2 u$ is

$$E_d(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{\imath}{4}H_0^{(1)}(\omega|\mathbf{x}-\mathbf{y}|), & (d=2), \\ \frac{e^{\imath\omega|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} & (d=3), \end{cases}$$

   where $\omega > 0$ is the frequency or wave number. Find $\nabla_{\mathbf{y}} E_d(\mathbf{x},\mathbf{y})$. Note that the two dimensional case uses one of the Hankel functions of the first kind.

3. (Section 1) Show that

$$(\mathrm{D}\psi)(\mathbf{z}) = \frac{\mathbf{c}\cdot\mathbf{z}}{|\mathbf{z}|^d} + \mathcal{O}(|\mathbf{z}|^{-d}), \qquad \text{where} \quad \mathbf{c} = \frac{1}{2^{d-1}\pi}\int_\Gamma \varphi(\mathbf{y})\boldsymbol{\nu}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}).$$

4. (Section 1) Let $u = \mathrm{S}\eta - \mathrm{D}\psi$. Give formulas for

$$\gamma^\pm u, \qquad \partial_\nu^\pm u, \qquad [\![\gamma u]\!], \qquad [\![\partial_\nu u]\!], \qquad \{\!\{\gamma u\}\!\}, \qquad \{\!\{\partial_\nu u\}\!\}.$$

5. (Section 1) Let us take a point $\mathbf{x}_0 \in \Omega_-$, and consider the function $u(\mathbf{x}) = 1/|\mathbf{x}-\mathbf{x}_0|$. Show that it is not possible to represent u as a double layer potential in Ω_+.

6. (Section 3) Give a direct boundary integral formulation based on Green's identity for the interior Dirichlet problem.

7. (Section 3) **The exterior Dirichlet-to-Neumann operator.** Consider the operator that given β_0 solves the exterior Dirichlet problem (2.18) and outputs $\mathrm{DtN}^+\beta_0 := \partial_\nu^+ u$.

   (a) Using (2.19)-(2.20) show that $\mathrm{DtN}^+ = \mathrm{V}^{-1}(-\frac{1}{2}\mathrm{I} + \mathrm{K})$.

   (b) Using now an indirect method, representing the solution of (2.18) in the form $u = \mathrm{S}\eta$, solve for η and then compute $\partial_\nu^+ u$ with help of the jump relations. Show that $\mathrm{DtN}^+ = (-\frac{1}{2}\mathrm{I} + \mathrm{K}^t)\mathrm{V}^{-1}$.

   (c) Use (a) and (b) to prove that $\mathrm{KV} = \mathrm{VK}^t$.

8. (Section 3) Following the ideas of the previous exercise, find two formulas for the interior Dirichlet-to-Neumann operator.

9. (Section 3) **Double layer potential and Dirichlet problem.** In the two dimensional case, to solve

$$\Delta u = 0 \quad \text{in } \Omega_+, \qquad \gamma^+ u = \beta_0, \qquad u = u_\infty + \mathcal{O}(r^{-1}),$$

   we try with a representation

$$u = \mathrm{D}\psi + u_\infty, \qquad \text{where} \quad \int_\Gamma \psi \, \mathrm{d}\Gamma = 0.$$

   Give a reason for the last condition. Write down the associated integral equation.

10. (Section 3) **Double layer potential and Dirichlet problem.** In the three dimensional case, to solve

$$\Delta u = 0 \quad \text{in } \Omega_+, \qquad \gamma^+ u = \beta_0, \qquad u = \mathcal{O}(r^{-1})$$

    where u_∞ is unknown, we try with a representation

$$u = \mathrm{D}\psi + c\Phi, \quad \text{where} \quad \int_\Gamma \psi \, \mathrm{d}\Gamma = 0, \quad \text{and} \quad \Phi(\mathbf{z}) = \frac{1}{|\mathbf{z} - \mathbf{x}_0|}, \qquad \mathbf{x}_0 \in \Omega_-.$$

    Give a reason for the condition on ψ and for the need to include the monopole Φ. Write down the associated integral equation.

Project # 2.1 – A Nyström method (coding)

Statement of the problem. This project can be considered as a continuation of Project # 1.1. We are looking for a numerical solution of the interior Dirichlet problem for the Laplacian

$$\Delta u = 0 \quad \text{in } \Omega_-, \qquad \gamma^- u = \beta_0,$$

where Γ can be parametrized using a 1-periodic function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ with the properties

$$\mathbf{x}(t+1) = \mathbf{x}(t) \quad \forall t, \qquad |\mathbf{x}'(t)| \neq 0 \quad \forall t, \qquad \mathbf{x}(t) \neq \mathbf{x}(\tau) \quad t - \tau \notin \mathbb{Z}.$$

The double layer representation and integral equation (2.23) will be given parametric forms using

$$\beta_0(t) := \beta_0(\mathbf{x}(t)), \qquad \psi(t) := \psi(\mathbf{x}(t)),$$

and a non-unit normal vector field

$$\mathbf{x}'(t)^\perp := (x_2'(t), -x_1'(t)),$$

assuming that the parametrization is positively oriented. The form of the potential is then

$$u(\mathbf{z}) = \int_0^1 \frac{(\mathbf{z} - \mathbf{x}(\tau)) \cdot \mathbf{x}'(\tau)^\perp}{2\pi |\mathbf{z} - \mathbf{x}(\tau)|^2} \psi(\tau) \mathrm{d}\tau,$$

and the integral equation is

$$-\frac{1}{2}\psi(t) - \int_0^1 \frac{(\mathbf{x}(t) - \mathbf{x}(\tau)) \cdot \mathbf{x}'(\tau)^\perp}{2\pi |\mathbf{x}(t) - \mathbf{x}(\tau)|^2} \psi(\tau) \mathrm{d}\tau = \beta_0(t) \qquad \forall t.$$

Show that

$$\lim_{t \to \tau} \frac{(\mathbf{x}(t) - \mathbf{x}(\tau)) \cdot \mathbf{x}'(\tau)^\perp}{2\pi |\mathbf{x}(t) - \mathbf{x}(\tau)|^2} = \frac{\mathbf{x}''(\tau) \cdot \mathbf{x}'(\tau)^\perp}{4\pi |\mathbf{x}'(\tau)|^2}.$$

A Nyström approximation. We then substitute integrals by composite trapezoidal approximations

$$\int_0^1 \phi(\tau) \approx \frac{1}{N} \sum_{j=1}^{N} \phi(t_j) = \frac{1}{N} (\tfrac{1}{2}\phi(t_0) + \sum_{j=1}^{N-1} \phi(t_j) + \tfrac{1}{2}\phi(t_N)) \qquad t_j = j/N.$$

(Recall that we are dealing with periodic functions.) Using this quadrature formula for the integral equation and the potential representation, it is simple to obtain a fully discrete approximation $u^N(\mathbf{z})$ for any $\mathbf{z} \in \Omega_-$. Check errors $|u^N(\mathbf{z}) - u(\mathbf{z})|$ for different geometries and exact solutions. You will see that the method converges extraordinarily fast.

Organization. One of the nices features of this simple method is the fact that all geometric information is contained in the quantities

$$\mathbf{m}_j := \mathbf{x}(t_j), \qquad \mathbf{n}_j := \tfrac{1}{N}\mathbf{x}'(t_j)^\perp, \qquad \ell_j := |\mathbf{x}'(t_j)|, \qquad \mathbf{s}_j := \mathbf{x}''(t_j),$$

and that the right-hand-side involves the evaluations of $\beta_0(\mathbf{x}(t_j)) = \beta_0(\mathbf{m}_j)$.

Project # 2.2 – A Galerkin direct method (coding)

General setting. This project is a continuation of Project # 1.2. Our goal is to code the Galerkin equations for a formulation similar to what is given in Section 2.3, in the two dimensional case. Let then Γ be a closed polygonal curve in the plane, discretized in elements $\{T_1, \ldots, T_N\}$. To solve the exterior Dirichlet problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega_+, \\ \gamma^+ u &= \beta_0 && (\text{on } \Gamma), \\ u &= u_\infty + \mathcal{O}(r^{-1}) && \text{as } r \to \infty, \end{aligned}$$

we use a direct formulation (Green's third identity) to represent the solution

$$u = \mathrm{D}\varphi - \mathrm{S}\lambda + u_\infty, \qquad \varphi = \beta_0, \qquad \lambda = \partial_\nu^+ u,$$

and derive an integral equation from one of Green's identities:

$$\mathrm{V}\lambda - u_\infty = -\tfrac{1}{2}\varphi + \mathrm{K}\varphi, \qquad \int_\Gamma \lambda \mathrm{d}\Gamma = 0.$$

Handling the discretization. We consider the two following discrete spaces:

$$\begin{aligned} X_h &:= \{\lambda^h : \Gamma \to \mathbb{R} \,:\, \lambda^h|_{T_i} \in \mathcal{P}_0(T_i) \quad \forall i\}, \\ Y_h &:= \{\varphi^h \in \mathcal{C}(\Gamma) \,:\, \varphi^h|_{T_i} \in \mathcal{P}_1(T_i) \quad \forall i\}. \end{aligned}$$

The basis functions for X_h are the characteristic functions of the elements: $\{\chi_1, \ldots, \chi_N\}$. There are N nodes, which are numbered in a separate way $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$, and they give rise to functions

$$\psi_i \in Y_h \qquad \text{s.t.} \quad \psi_i(\mathbf{v}_j) = \delta_{ij}.$$

The discrete equations. The Galerkin equations are

$$\sum_j V_{ij}\lambda_j - u_\infty^h |T_i| = -\tfrac{1}{2}\sum_j M_{ij}\varphi_j + \sum_j K_{ij}\varphi_j, \qquad \forall i, \qquad \sum_j |T_j|\lambda_j = 0$$

(see Chapter 1 and in particular Project # 1.1). The computation of V_{ij} and the way of storing geometric information is explained in Project # 1.1.

Assembly in Y_h. We now explain how to compute integrals of the following form

$$I_j := \int_\Gamma \phi(\mathbf{y})\psi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \qquad j = 1, \ldots, N. \tag{2.29}$$

Assume that for the element $T_\ell = [\mathbf{v}_{\mathrm{ele}(\ell,1)}, \mathbf{v}_{\mathrm{ele}(\ell,2)}]$ we can find a function ϕ_ℓ such that

$$\phi_\ell \in \mathcal{P}_1(\Gamma_\ell), \qquad \phi_\ell(\mathbf{v}_{\mathrm{ele}(\ell,1)}) = 1, \qquad \phi_\ell(\mathbf{v}_{\mathrm{ele}(\ell,2)}) = 0,$$

and that we have computed the quantities

$$I_\ell^{(1)} = \int_{T_\ell} \phi(\mathbf{y})\phi_\ell(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \qquad I_\ell^{(2)} = \int_{T_\ell} \phi(\mathbf{y})(1-\phi_\ell(\mathbf{y}))\mathrm{d}\Gamma(\mathbf{y}).$$

The assembly process is a loop that after initializing $I_j = 0$ for all j, computes

$$I_{\mathrm{ele}(j,1)} = I_{\mathrm{ele}(j,1)} + I_j^{(1)}, \qquad I_{\mathrm{ele}(j,2)} = I_{\mathrm{ele}(j,2)} + I_j^{(2)}, \qquad j = 1, \ldots, N.$$

This process transfers computations done at the element level, and numbered with element indices, to computations on the entire boundary, counted by vertices.

Applications. Since

$$\int_{T_j} \phi_\ell(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \int_{T_j} (1-\phi_\ell(\mathbf{y}))\mathrm{d}\Gamma(\mathbf{y}) = \tfrac{1}{2}|T_\ell|\delta_{i\ell},$$

it is easy to compute now

$$M_{ij} = \int_{T_i} \psi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma \chi_i(\mathbf{y})\psi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \qquad \forall i,j.$$

To compute the matrix related to the double layer operator, we approximate

$$\begin{aligned} K_{ij} &= \int_{T_i}\int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2\pi|\mathbf{x}-\mathbf{y}|^2}\psi_i(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ &\approx \frac{|T_i|}{2}\Big(\int_\Gamma \frac{(\mathbf{g}_i^+-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2\pi|\mathbf{g}_i^+-\mathbf{y}|^2}\psi_i(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) + \frac{(\mathbf{g}_i^--\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{2\pi|\mathbf{g}_i^--\mathbf{y}|^2}\psi_i(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})\Big), \end{aligned}$$

where the two Gaussian points on the element T_i are defined in Project # 1.2. The above integrals can be computed exactly (see what follows), after some element-by-element computations and assembly process.

An exact computation. Let $e = [\mathbf{v}, \mathbf{w}]$ be a segment, let $\mathbf{z} \in \mathbb{R}^2$, and let

$$\begin{aligned} \ell &:= |\mathbf{w}-\mathbf{v}|, & \eta &:= (\mathbf{z}-\mathbf{v})\cdot\mathbf{n}, \\ \mathbf{t} &:= \frac{1}{\ell}(\mathbf{w}-\mathbf{v}), & \alpha &:= -(\mathbf{z}-\mathbf{v})\cdot\mathbf{t}, \\ \mathbf{n} &:= \mathbf{t}^\perp = (t_2, -t_1), & \beta &:= \alpha+\ell = -(\mathbf{z}-\mathbf{w})\cdot\mathbf{t}. \end{aligned}$$

Let also $\phi \in \mathcal{P}_1(e)$ satisfy $\phi(\mathbf{v}) = 1$, $\phi(\mathbf{w}) = 0$. Then

$$\begin{aligned} \int_e \frac{(\mathbf{z}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2}\begin{bmatrix}\phi(\mathbf{y})\\ 1-\phi(\mathbf{y})\end{bmatrix}\mathrm{d}\Gamma(\mathbf{y}) &= \frac{1}{\ell}\left(\arctan\left(\frac{\beta}{\eta}\right) - \arctan\left(\frac{\alpha}{\eta}\right)\right)\begin{bmatrix}\beta\\ -\alpha\end{bmatrix} \\ &\quad + \frac{\eta}{2\ell}\log\frac{\eta^2+\beta^2}{\eta^2+\alpha^2}\begin{bmatrix}-1\\ 1\end{bmatrix} \end{aligned}$$

Chapter 3

The Calderón Calculus

chap:3

In this chapter we repeat the presentation of the two layer potentials and four integral operators associated to the Laplace equation and give arguments for some of their properties. The point of view is to ignore what the potentials are, and try to work with them for what properties they satisfy. The presentation is not fully rigorous (function spaces are still missing), but the arguments that we show here will be the ones used in the rigorous proofs in forthcoming chapters. In order to prepare the mindset for this chapter, let us mention that all the presentation hinges on two important results, with the minor intrusion of an energy concept:

- We will assume that a certain transmission problem in free space has a unique solution. We will use it to define the potentials and their associated boundary integral operators.
- We will assume an integration by parts formula, which we will use to show some symmetries of the integral operators, as well as some positivity results.
- Finally, we will have a look at the concept of energy that is associated to the integration by parts formula. This concept will deliver the energy-free solutions that create some uniqueness issues.

3.1 Layer potentials and transmission problems

sec:3.1
eq:3.1

The transmission problem. Let us assume that for given $\eta, \psi : \Gamma \to \mathbb{R}$, the problem

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{3.1a}$$
$$[\![\gamma u]\!] = \psi \qquad (\text{on } \Gamma), \tag{3.1b}$$
$$[\![\partial_\nu u]\!] = \eta \qquad (\text{on } \Gamma), \tag{3.1c}$$
$$u = \mathcal{O}(r^{-1}) \qquad \text{at infinity}, \tag{3.1d}$$

admits a unique solution. This fact includes the additional hypothesis of data

$$\int_\Gamma \eta \, \mathrm{d}\Gamma = 0 \qquad \text{when } d = 2. \tag{3.2}$$

eq:3.2

The solution of (3.1) with $\psi = 0$ is denoted $u = \mathrm{S}\eta$. The solution of (3.1) with $\eta = 0$ is denoted $u = \mathrm{D}\psi$. Because of the superposition principle (all operators in (3.1) are linear and we are assuming existence and uniqueness of solution), we can write the solution to (3.1) as

$$u = \mathrm{S}\eta - \mathrm{D}\psi.$$

Let us briefly repeat these definitions step by step.

eq:3.3

Given η satisfying (3.2), the single layer potential $u = \mathrm{S}\eta$ is the solution to:

$$\begin{aligned} \Delta u &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, && (3.3a)\\ [\![\gamma u]\!] &= 0 && (\text{on } \Gamma), && (3.3b)\\ [\![\partial_\nu u]\!] &= \eta && (\text{on } \Gamma), && (3.3c)\\ u &= \mathcal{O}(r^{-1}) && \text{at infinity}, && (3.3d) \end{aligned}$$

eq:3.4

Given ψ, the double layer potential $u = \mathrm{D}\psi$ is the solution to:

$$\begin{aligned} \Delta u &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, && (3.4a)\\ [\![\gamma u]\!] &= -\psi && (\text{on } \Gamma), && (3.4b)\\ [\![\partial_\nu u]\!] &= 0 && (\text{on } \Gamma), && (3.4c)\\ u &= \mathcal{O}(r^{-1}) && \text{at infinity}, && (3.4d) \end{aligned}$$

Energy-free solutions. The energy associated to a function is the quantity

$$E(u) := \tfrac{1}{2} \int_{\mathbb{R}^d \setminus \Gamma} |\nabla u|^2.$$

If $E(u) = 0$ and $u = \mathcal{O}(r^{-1})$ at infinity, then it is clear that $u \in \operatorname{span}\{\chi_{\Omega_-}\}$. These functions are the energy free solutions of the problem. They are related to the layer potentials in the following way. It is clear that $\Delta\chi_{\Omega_-} = 0$ in $\mathbb{R}^d \setminus \Gamma$, that $[\![\chi_{\Omega_-}]\!] = 1$ and $[\![\gamma\partial_\nu\chi_{\Omega_-}]\!] = 0$. Therefore

$$\mathrm{D}1 = -\chi_{\Omega_-}. \tag{3.5}$$

Jumps. By definition of the layer potentials through problem (3.1), it follows that

$$[\![\gamma \mathrm{S}\eta]\!] = 0, \qquad [\![\partial_\nu \mathrm{S}\eta]\!] = \eta, \qquad [\![\gamma \mathrm{D}\psi]\!] = -\psi, \qquad [\![\partial_\nu \mathrm{D}\psi]\!] = 0.$$

We can write these identities in operator form:

$$\begin{bmatrix} [\![\gamma\cdot]\!] \\ [\![\partial_\nu\cdot]\!] \end{bmatrix} \begin{bmatrix} -\mathrm{D} & \mathrm{S} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} \end{bmatrix} =: \mathbb{I}. \tag{3.6}$$ eq:3.6

Averages. The four boundary integral operators are defined as averages of side values of the layer potentials:

$$\begin{aligned} \mathrm{V}\eta &=: \{\!\{\gamma \mathrm{S}\eta\}\!\} = \gamma^{\pm}\mathrm{S}\eta, \\ \mathrm{K}\psi &=: \{\!\{\gamma \mathrm{D}\psi\}\!\}, \\ \mathrm{K}^t\eta &=: \{\!\{\partial_\nu \mathrm{S}\eta\}\!\}, \\ \mathrm{W}\psi &=: -\{\!\{\partial_\nu \mathrm{D}\psi\}\!\} = -\partial_\nu^{\pm}\mathrm{D}\psi. \end{aligned}$$

Mimicking (3.6), we can write

$$\begin{bmatrix} \{\!\{\gamma\cdot\}\!\} \\ \{\!\{\partial_\nu\cdot\}\!\} \end{bmatrix} \begin{bmatrix} -\mathrm{D} & \mathrm{S} \end{bmatrix} = \begin{bmatrix} -\mathrm{K} & \mathrm{V} \\ \mathrm{W} & \mathrm{K}^t \end{bmatrix} =: \mathbb{D}, \tag{3.7}$$ eq:3.7

understading now that these equalities are the definitions of the operators on the right-hand-side. From (3.6) and (3.7), it is also clear that

$$\partial_\nu^{\pm}\mathrm{S} = \mp\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t, \qquad \gamma^{\pm}\mathrm{D} = \pm\tfrac{1}{2}\mathrm{I} + \mathrm{K}.$$

The representation theorem. Let u satisfy

$$\Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity.} \tag{3.8}$$ eq:3.8

Then

$$u = \mathrm{S}[\![\partial_\nu u]\!] - \mathrm{D}[\![\gamma u]\!]. \tag{3.9}$$ eq:3.9

The reason for this to hold is really simple: let $\eta := [\![\partial_\nu u]\!]$ and $\psi := [\![\gamma u]\!]$. Clearly u and $\mathrm{S}\eta - \mathrm{D}\psi$ are solutions to (3.1); therefore, they are equal. Implicit to this argument is the fact that if u satisfies (3.8) and we are in two dimensions, then $\int_\Gamma [\![\partial_\nu u]\!] \mathrm{d}\Gamma = 0$.

3.2 The exterior Calderón projector

Representation theorem for exterior solutions. Let

$$\Delta u = 0 \quad \text{in } \Omega_+, \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity.} \tag{3.10}$$ eq:3.10

We then extend u by zero to Ω_-:

$$\widetilde{u} = \begin{cases} u & \text{in } \Omega_+, \\ 0 & \text{in } \Omega_-, \end{cases} \tag{3.11}$$ eq:3.11

and apply the representation formula (3.9) to $\widetilde{u}$:

$$\widetilde{u} = \mathrm{S}[\![\partial_\nu \widetilde{u}]\!] - \mathrm{D}[\![\gamma \widetilde{u}]\!] = -\mathrm{S}\partial_\nu^{+} u + \mathrm{D}\gamma^{+} u. \tag{3.12}$$ eq:3.12

We can rephrase this formula forgetting about the extended function $\widetilde{u}$, by simply writing

$$\mathrm{D}\gamma^{+} u - \mathrm{S}\partial_\nu^{+} u = \begin{cases} u & \text{in } \Omega_+, \\ 0 & \text{in } \Omega_-. \end{cases}$$

Integral identities. There are different ways of reaching the two integral identities associated to the limiting values of an exterior solution of the Laplace equation (3.10). We can think in terms of the extended function (3.12) to obtain the identities

$$\begin{aligned}\tfrac{1}{2}\gamma^+ u =& \{\!\{\gamma \widetilde{u}\}\!\} &= -\mathrm{V}\partial_\nu^+ u + \mathrm{K}\gamma^+ u,\\ \tfrac{1}{2}\partial_\nu^+ u =& \{\!\{\partial_\nu \widetilde{u}\}\!\} &= -\mathrm{K}^t\partial_\nu^+ u - \mathrm{W}\gamma^+ u.\end{aligned}$$

These equalities follow from (3.12) and the definition of the integral operators as averages (3.7). Also, we can start with the identity $u = \mathrm{D}\gamma^+ u - \mathrm{S}\partial_\nu^+ u$, valid in Ω_+, take exterior values from both sides, and show that

$$\begin{aligned}\gamma^+ u &= \gamma^+\mathrm{D}\gamma^+ u - \gamma^+\mathrm{S}\partial_\nu^+ u = \tfrac{1}{2}\gamma^+ u + \mathrm{K}\gamma^+ u - \mathrm{V}\partial_\nu^+ u\\ \partial_\nu^+ u &= \partial_\nu^+\mathrm{D}\gamma^+ u - \partial_\nu^+\mathrm{S}\partial_\nu^+ u = -\mathrm{W}\gamma^+ u + \tfrac{1}{2}\partial_\nu^+ u - \mathrm{K}^t\partial_\nu^+ u.\end{aligned}$$

For easy reference, let us write the formulas again, after changing some signs:

$$\mathrm{V}\partial_\nu^+ u + (\tfrac{1}{2}\mathrm{I} - \mathrm{K})\gamma^+ u = 0. \tag{3.13a}$$

$$(\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t)\partial_\nu^+ u + \mathrm{W}\gamma^+ u = 0. \tag{3.13b}$$

The Calderón projector. We next use the formula

$$\begin{bmatrix} \gamma^+ \\ \partial_\nu^+ \end{bmatrix} = -\tfrac{1}{2}\begin{bmatrix} [\![\gamma\cdot]\!] \\ [\![\partial_\nu\cdot]\!] \end{bmatrix} + \begin{bmatrix} \{\!\{\gamma\cdot\}\!\} \\ \{\!\{\partial_\nu\cdot\}\!\} \end{bmatrix}$$

together with (3.6) and (3.7) to show that

$$\begin{bmatrix} \gamma^+ \\ \partial_\nu^+ \end{bmatrix}\begin{bmatrix} \mathrm{D} & -\mathrm{S} \end{bmatrix} = \tfrac{1}{2}\mathbb{I} - \mathbb{D} = \begin{bmatrix} \frac{1}{2}\mathrm{I} + \mathrm{K} & -\mathrm{V} \\ -\mathrm{W} & \frac{1}{2}\mathrm{I} - \mathrm{K}^t \end{bmatrix}, \tag{3.14}$$

eq:3.14

where we have changed signs in front of single and double layer potentials due to the change difference between the representation formula for transmission problems (3.9) and for exterior problems (3.12). The matrix of operators

$$\mathbb{C}_+ := \begin{bmatrix} \frac{1}{2}\mathrm{I} + \mathrm{K} & -\mathrm{V} \\ -\mathrm{W} & \frac{1}{2}\mathrm{I} - \mathrm{K}^t \end{bmatrix} \tag{3.15}$$

will be called the *exterior Calderón projector*, for reasons that we will next see.

The Calderón projector is a projector. We have already seen that if u is an exterior solution of the Laplacian (3.10), then

$$\mathbb{C}_+\begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix}.$$

This means that elements of the set

$$\mathcal{C} := \{(\gamma^+ u, \partial_\nu^+ u) \,:\, u \text{ satisfies (3.10)}\}, \tag{3.16}$$

i.e., *Cauchy data* of solutions of the exterior Laplace equation, are fixed points of $\mathbb{C}_+$. Reciprocally, if

$$\mathbb{C}_+ \begin{bmatrix} \psi \\ \eta \end{bmatrix} = \begin{bmatrix} \psi \\ \eta \end{bmatrix},$$

we can define $u = \mathrm{D}\psi - \mathrm{S}\eta$, which is a solution of (3.10) and note that

$$\begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \begin{bmatrix} \gamma^+ \\ \partial_\nu^+ \end{bmatrix} \begin{bmatrix} \mathrm{D} & -\mathrm{S} \end{bmatrix} \begin{bmatrix} \psi \\ \eta \end{bmatrix} = \mathbb{C}_+ \begin{bmatrix} \psi \\ \eta \end{bmatrix} = \begin{bmatrix} \psi \\ \eta \end{bmatrix},$$

which shows that the fixed points of $\mathbb{C}_+$ are the elements of $\mathcal{C}$. We can think of this in a slightly different way: we start with general densities (ψ, η), define the potential $u = \mathrm{D}\psi - \mathrm{S}\eta$ and note that

$$\mathbb{C}_+ \begin{bmatrix} \psi \\ \eta \end{bmatrix} = \begin{bmatrix} \gamma^+ \\ \partial_\nu^+ \end{bmatrix} \begin{bmatrix} \mathrm{D} & -\mathrm{S} \end{bmatrix} \begin{bmatrix} \psi \\ \eta \end{bmatrix} = \begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \mathbb{C}_+ \begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \mathbb{C}_+ \mathbb{C}_+ \begin{bmatrix} \psi \\ \eta \end{bmatrix}.$$

The reader should by now be convinced that we have shown the following two properties:

$$\mathbb{C}_+^2 = \mathbb{C}_+ \quad \text{and} \quad \operatorname{range} \mathbb{C}_+ = \{(\gamma^+ u, \partial_\nu^+ u) \,:\, u \text{ satisfies (3.10)}\}, \tag{3.17}$$

or in words, $\mathbb{C}_+$ is a projection onto the space of Cauchy data of exterior solutions of the Laplace equation.

Some identities between the operators. We are now going to exploit the projection property of $\mathbb{C}_+$. To do this, just note that

$$\begin{aligned} \mathbb{C}_+\mathbb{C}_+ = \mathbb{C}_+ \qquad &\Longleftrightarrow \qquad (\tfrac{1}{2}\mathbb{I} - \mathbb{D})(\tfrac{1}{2}\mathbb{I} - \mathbb{D}) = \tfrac{1}{2}\mathbb{I} - \mathbb{D} \\ &\Longleftrightarrow \qquad \tfrac{1}{4}\mathbb{I} - \mathbb{D} + \mathbb{D}^2 = \tfrac{1}{2}\mathbb{I} - \mathbb{D} \\ &\Longleftrightarrow \qquad \mathbb{D}^2 = \tfrac{1}{4}\mathbb{I}, \end{aligned}$$

or in expanded form,

$$\begin{bmatrix} \mathrm{K}^2 + \mathrm{VW} & -\mathrm{KV} + \mathrm{VK}^t \\ -\mathrm{WK} + \mathrm{K}^t\mathrm{W} & \mathrm{WV} + (\mathrm{K}^t)^2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} \end{bmatrix}. \tag{3.18}$$

This matrix equality can also be written as a collection of four identities:

$$\mathrm{VW} = \tfrac{1}{2}\mathrm{I} - \mathrm{K}^2, \qquad \mathrm{WV} = \tfrac{1}{4}\mathrm{I} - (\mathrm{K}^t)^2, \tag{3.19a}$$

$$\mathrm{KV} = \mathrm{VK}^t, \qquad \mathrm{WK} = \mathrm{K}^t\mathrm{W}. \tag{3.19b}$$

Some readers might recognize one of these formulas from the exercise list in Chapter 2. We got to it from apparently different arguments.

3.3 The integration by parts formula

Notation. In order to avoid an excess of integral signs (and in preview of some inner product style notation for duality products), we will write for functions u, v and vector fields $\mathbf{p}, \mathbf{q}$ (all of them defined on a domain Ω)

$$(u, v)_\Omega := \int_\Omega u\, v, \qquad (\mathbf{p}, \mathbf{q})_\Omega := \int_\Omega \mathbf{p} \cdot \mathbf{q}.$$

We will also shorten

$$\|u\|^2_\Omega := (u, u)_\Omega, \qquad \|\mathbf{p}\|^2_\Omega := (\mathbf{p}, \mathbf{p})_\Omega.$$

For functions defined on Γ, we denote

$$\langle \eta, \phi \rangle_\Gamma := \int_\Gamma \eta\, \phi \, \mathrm{d}\Gamma.$$

Even if at the present time, this latter integral symbol is just the $L^2(\Gamma)$ inner product, whenever we find an expression of the kind $\langle \partial_\nu u, \gamma v \rangle_\Gamma$, which later on will be made correspond to a duality product, we will always place normal derivatives in the first variable.

Around Green's First Identity. The integration by parts formula we will consider is a variant of Green's First Identity, written simultaneously for the domains $\Omega_\pm$: if

$$\Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \qquad \text{and} \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity}, \tag{3.20}$$

eq:3.20

then

$$(\nabla u, \nabla v)_{\mathbb{R}^d \setminus \Gamma} = \langle \partial^-_\nu u, \gamma^- v \rangle_\Gamma - \langle \partial^+_\nu u, \gamma^+ v \rangle_\Gamma. \tag{3.21}$$

Note how this is formula (2.8) especialized to a harmonic function u, applied in $\Omega_\pm$. The reason for the negative sign in the term coming from the exterior lies on the sign of the normal vector field, which is pointing inwards from the point of view of Ω_+. Very simple algebra allows us to write

$$(\nabla u, \nabla v)_{\mathbb{R}^d \setminus \Gamma} = \langle [\![\partial_\nu u]\!], \gamma^- v \rangle_\Gamma + \langle \partial^+_\nu u, [\![\gamma v]\!] \rangle_\Gamma. \tag{3.22}$$

eq:3.22

If we now aply (3.22) to two functions satisfying (3.20), we obtain

$$\langle [\![\partial_\nu u]\!], \gamma^- v \rangle_\Gamma + \langle \partial^+_\nu u, [\![\gamma v]\!] \rangle_\Gamma = (\nabla u, \nabla v)_{\mathbb{R}^d \setminus \Gamma} = \langle [\![\partial_\nu v]\!], \gamma^- u \rangle_\Gamma + \langle \partial^+_\nu v, [\![\gamma u]\!] \rangle_\Gamma. \tag{3.23}$$

eq:3.23

Symmetry and positivity of V. Apply now (3.23) to $u = \mathrm{S}\eta$ and $v = \mathrm{S}\mu$. Then

$$\langle \eta, \mathrm{V}\mu \rangle_\Gamma = \langle [\![\partial_\nu u]\!], \gamma v \rangle_\Gamma = (\nabla u, \nabla v)_{\mathbb{R}^d \setminus \Gamma} = \langle [\![\partial_\nu v]\!], \gamma u \rangle_\Gamma = \langle \mu, \mathrm{V}\eta \rangle_\Gamma,$$

which, in formal notation for transposition with respect to the $\langle \cdot, \cdot \rangle_\Gamma$ bracket, can be shortened as $\mathrm{V}^t = \mathrm{V}$. Moreover, if $\eta \neq 0$,

$$\langle \eta, \mathrm{V}\eta \rangle_\Gamma = \|\nabla u\|^2_{\mathbb{R}^d \setminus \Gamma} > 0, \tag{3.24}$$

eq:3.24

since the only energy-free solution of the Laplace equation (see Section 3.1) cannot be represented as a single layer potential, because it has to jump across Γ.

Symmetry and positivity of W. Applying (3.23) to $u = \mathrm{D}\psi$ and $v = \mathrm{D}\phi$, we obtain

$$\langle \mathrm{W}\psi, \phi\rangle_\Gamma = \langle -\partial_\nu u, -[\![\gamma v]\!]\rangle_\Gamma = (\nabla u, \nabla v)_{\mathbb{R}^d\setminus\Gamma} = \langle -\partial_\nu v, -[\![\gamma u]\!]\rangle_\Gamma = \langle \mathrm{W}\phi, \psi\rangle_\Gamma,$$

or $\mathrm{W}^t = \mathrm{W}$. The positivity result is not as strong as in the case of the operator V. If $u = \mathrm{D}\psi$, then

$$\langle \mathrm{W}\psi, \psi\rangle_\Gamma = \|\nabla u\|^2_{\mathbb{R}^d\setminus\Gamma} \geq 0. \tag{3.25}$$

eq:3.25

The gradient is equal to zero only if u is a multiple of $\chi_{\Omega_-} = -\mathrm{D}1$, that is, only when $\psi \in \mathcal{P}_0(\Gamma)$. Note that both (3.24) and (3.25) will be strenghthened to coercivity inequalities in its due time.

An orthogonality result. Take finally $u = \mathrm{S}\eta$ and $v = \mathrm{D}\psi$ in (3.23). Then

$$\langle [\![\partial_\nu u]\!], \gamma^- v\rangle_\Gamma + \langle \partial_\nu^+ u, [\![\gamma v]\!]\rangle_\Gamma = (\nabla u, \nabla v)_{\mathbb{R}^d\setminus\Gamma} = 0.$$

This proves that single and double layer potentials are orthogonal with respect to the semi-inner product $(\nabla\cdot, \nabla\cdot)_{\mathbb{R}^d\setminus\Gamma}$. Moreover, we can write this equality in terms of the densities as

$$\langle \eta, -\tfrac{1}{2}\psi + \mathrm{K}\psi\rangle_\Gamma + \langle \tfrac{1}{2}\eta + \mathrm{K}^t\eta, -\psi\rangle_\Gamma = 0,$$

which is equivalent to

$$\langle \eta, \mathrm{K}\psi\rangle_\Gamma = \langle \mathrm{K}^t\eta, \psi\rangle_\Gamma,$$

and thus justifies the notation chosen for K^t. It is important to emphasize that the three transposition properties we have displayed are given using just this formal definitions of the potentials as solutions of transmission problems. Two of them (the symmetry of V and the fact that K^t is the transpose of K) could be seen directly from the integral definitions of these operators.

3.4 Literature, exercises, and working projects

1. (Section 1) Show that

$$\mathrm{W}1 = 0 \qquad \text{and} \qquad \mathrm{K}1 = -\tfrac{1}{2}.$$

2. (Section 2) Show that the kernel of the matrix of operators

$$\begin{bmatrix} \mathrm{V} & \frac{1}{2}\mathrm{I} - \mathrm{K}^t \\ \frac{1}{2}\mathrm{I} + \mathrm{K} & \mathrm{W} \end{bmatrix}$$

is the space $\{(\partial_\nu^+ u, \gamma^+ u) \,:\, \Delta u = 0 \text{ in } \Omega_+, u = \mathcal{O}(r^{-1}) \text{ at infinity}\}$.

3. (Section 2) **The interior Calderón projector.** Let

$$\mathbb{C}_- := \begin{bmatrix} \gamma^- \\ \partial_\nu^- \end{bmatrix} \begin{bmatrix} -\mathrm{D} & \mathrm{S} \end{bmatrix}.$$

Identify the elements of $\mathbb{C}_-$, show that

$$\mathbb{C}_+ + \mathbb{C}_- = \mathbb{I}$$

and conclude that $\mathbb{C}_-$ and $\mathbb{C}_+$ are complementary projections. What is the range of $\mathbb{C}_-$?

4. (Section 2) **Combined field operators.** Consider the followin combined field potential:

$$u := (\mathrm{S} + c\,\mathrm{D})\eta,$$

where $c \in \mathbb{R}$. What are the values of $\gamma^+ u$ and $\partial_\nu^+ u$?

5. (Section 2) **Burton-Miller type equation.** Let $(\beta_0, \beta_1) := (\gamma v, \partial_\nu v)$ be the Cauchy data on Γ of a solution of $\Delta v = 0$ in a neighborhood of Ω_-. (This implies that $\Delta v = 0$ in Ω_- and that the interior and exterior limits of v coincide.) Let u satisfy

$$\Delta u = 0 \quad \text{in } \Omega_+, \qquad \gamma^+ u + \beta_0 = 0, \qquad u = \mathcal{O}(r^{-1}) \quad \text{at infinity.}$$

Show that for any constant $c \neq 0$

$$(\mathrm{V} + c(\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t))\partial_\nu^+ u = -((-\tfrac{1}{2}\mathrm{I} + \mathrm{K}) - c\mathrm{W})\beta_0.$$

Show that

$$(\mathrm{V} + c(\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t))(\partial_\nu^+ + \beta_1) = \beta_0 + c\beta_1.$$

6. (Section 3) Show that KV and WK are formally symmetric.

7. (Section 3) Consider the matrix of operators

$$\mathbb{M}_+ := \begin{bmatrix} \mathrm{V} & -\frac{1}{2}\mathrm{I} - \mathrm{K}^t \\ \frac{1}{2}\mathrm{I} + \mathrm{K} & \mathrm{W} \end{bmatrix}.$$

Show that it is formally positive semidefinite and that its kernel is $\{0\} \times \mathcal{P}_0(\Gamma)$. If u is an exterior solution of the Laplacian, find what

$$\mathbb{M}_+ \begin{bmatrix} \partial_\nu^+ u \\ \gamma^+ u \end{bmatrix}$$

is.

Project # 3.1 – Linear elasticity

The basic concepts. In linear elasticity the main unknown is the displacement field $\mathbf{u}$. Associated to it we have strain

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \tfrac{1}{2}(\mathbf{D}\mathbf{u} + (\mathbf{D}\mathbf{u})^\top) \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and stress

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\mathbf{I}, \qquad \sigma_{ij}(\mathbf{u}) = 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\delta_{ij}.$$

The constants λ, μ are assumed to be positive. Instead of the normal derivative, we consider the normal stress (or traction) as a boundary relevant quantity $\boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu}$. The Navier-Lamé equation in absence of volumetric forces is

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}.$$

The energy function is

$$(\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u}))_\Omega = \sum_{i,j=1^d} (\sigma_{ij}(\mathbf{u}), \varepsilon_{ij}(\mathbf{u}))_\Omega = 2\mu(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u}))_\Omega + \lambda\|\operatorname{div}\mathbf{u}\|^2_\Omega.$$

Potentials. We assume that the transmission problem

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{0} && \text{in } \mathbb{R}^d \setminus \Gamma, \\ [\![\gamma\mathbf{u}]\!] &= \boldsymbol{\psi} && (\text{on } \Gamma), \\ [\![\boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu}]\!] &= \boldsymbol{\eta} && (\text{on } \Gamma), \\ \mathbf{u} &= \mathcal{O}(r^{-1}) && \text{at infinity}, \end{aligned}$$

is uniquely solvable, with the additional restriction

$$\int_\Gamma \boldsymbol{\eta}\, \mathrm{d}\Gamma = \mathbf{0} \qquad \text{when } d = 2.$$

Potentials and operators are defined using the same strategy as in the case of the Laplace equation. The associated energy-free solutions are of the form

$$\chi_{\Omega_-}\mathbf{m}, \qquad \mathbf{m} \in \mathcal{M}_d,$$

where the space of rigid motions depends on the space dimension. Its elements are

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{if } d = 2,$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & -b_2 \\ b_1 & 0 & b_3 \\ b_2 & -b_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \text{if } d = 3.$$

Integration by parts. If $\mathrm{div}\boldsymbol{\sigma}(\mathbf{u}) = 0$ in $\mathbb{R}^d \setminus \Gamma$ and $\mathbf{u} = \mathcal{O}(r^{-1})$ at infinity, then

$$(\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathbb{R}^d \setminus \Gamma} = \langle \boldsymbol{\sigma}(\mathbf{u})^+ \boldsymbol{\nu}, \gamma^+ \mathbf{v} \rangle_\Gamma - \langle \boldsymbol{\sigma}(\mathbf{u})^- \boldsymbol{\nu}, \gamma^- \mathbf{v} \rangle_\Gamma.$$

Going step by step, derive the entire Calderón Calculus for the homogeneous isotropic linear elasticity operator.

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