Typos

1. In Page 42, there's a $\mathcal{L}(X,Y)$ that should be $\mathcal{B}(X,Y)$.

Errata

1. The first mistake is quite stupid and easy to catch:

In Proposition 3.1.2,

$$k := \max\{0, \lfloor \mu + 2 \rfloor\}$$

2. The second one is trickier and it corresponds to a mistake in a proof (thanks to Maryna Kachanovska for catching the mistake). The result is taken from reference [43] where the proof is also wrong but in a different way.

In Proposition 3.2.2, we have to eliminate the factor 2^μ in the right hand side of (3.7) and change the function C_ε to

$$C_\varepsilon(t) := \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+1)/2)} \frac{t^\varepsilon}{(1+t)^\varepsilon}.$$

The proof is based on the following corrected estimate

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|\sigma + \imath\omega|^\mu}{|1+\sigma+\imath\omega|^2} \mathrm{d}\omega &\le \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{((1+\sigma)^2+\omega^2)^{1-\mu/2}} \\ &= \frac{1}{(1+\sigma)^\varepsilon} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(1+\omega^2)^{(\varepsilon+1)/2}} \\ &= \frac{1}{(1+\sigma)^\varepsilon} \sqrt{\pi} \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+1)/2)}.\end{aligned}$$

Everything else in the original proof is correct. Note that, since we never show explicit dependence of the constants with respect to ε (or μ), all uses of this theorem are still correct. (An improvement over this result is given in the next section of this document.)

Improvements

Proposition 3.2.2. *Let* $\mathrm{A} = \mathcal{L}\{a\} \in \mathcal{A}(m+\mu, \mathcal{B}(X,Y))$ *with a non-negative integer* m *and* $\mu \in [0,1)$ *and let* $\varepsilon := 1-\mu \in (0,1]$*. If* $g \in \mathcal{C}^{m+1}(\mathbb{R}, X)$ *is causal and* $g^{(m+2)}$ *is integrable, then* $a * g \in \mathcal{C}(\mathbb{R}, Y)$ *is causal and*

$$\|(a * g_(t)\| \le C_\varepsilon(t) C_{\mathrm{A}}(t^{-1}) \int_0^t \|\mathcal{P}_2 g^{(m)}(\tau)\| \mathrm{d}\tau,$$

where

$$C_\varepsilon(t) := \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+1)/2)} \left(\frac{t}{1+t}\right)^\varepsilon.$$

and

$$\mathcal{P}_2 g = g + 2\dot{g} + \ddot{g}.$$

Proof. We first prove the result for $m = 0$. If $\ddot{g}$ is causal and integrable, the it is tempered. Furthermore, it is the derivative of a continuous bounded causal function, and therefore $\ddot{g} \in \mathrm{TD}(X)$. Moreover, $\mathcal{L}\{\mathcal{P}_2 g\} = (1+s)^2 \mathrm{G}(s) \in \mathcal{A}(0, X)$ (see Proposition 3.1.3) and therefore $\mathrm{G} \in \mathcal{A}(-2, X)$ and

$$\|(1+s)^2 \mathrm{G}(s)\| \le \int_0^\infty \|(\mathcal{P}_2 g)(\tau)\| \mathrm{d}\tau \qquad \forall s \in \mathbb{C}_+.$$

A simple bound now shows that $\mathrm{AF} \in \mathcal{A}(\mu - 2, Y)$ and because $\mu - 2 = -(1+\varepsilon) < -1$, it follows from Proposition 3.1.1 that $a * g$ is continuous and causal. We can bound $(a * g)(t)$ using the strong form of the inversion formula proceeding as follows:

$$\begin{aligned}
\|(a*g)(t)\| &\le \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^\infty \|\mathrm{A}(\sigma + \imath\omega)\mathrm{G}(\sigma + \imath\omega)\| \mathrm{d}\omega && \text{(by (3.2))} \\
&\le \frac{e^{\sigma t}}{2\pi} C_{\mathrm{A}}(\sigma) \int_{-\infty}^\infty \|(\sigma + \imath\omega)^\mu \mathrm{G}(\sigma + \imath\omega)\| \mathrm{d}\omega \\
&\le \frac{e^{\sigma t}}{2\pi} C_{\mathrm{A}}(\sigma) \max_{\mathrm{Re}\, s = \sigma} \{(1+s)^2 \mathrm{G}(s)\| \int_{-\infty}^\infty \frac{|\sigma + \imath\omega|^\mu}{|1 + \sigma + \imath\omega|^2} \mathrm{d}\omega \\
&\le \frac{e^{\sigma t}}{2\pi} C_{\mathrm{A}}(\sigma) \int_{-\infty}^\infty \frac{|\sigma + \imath\omega|^\mu}{|1 + \sigma + \imath\omega|^2} \mathrm{d}\omega \int_0^\infty \|(\mathcal{P}_2 g)(\tau)\| \mathrm{d}\tau. && \text{(see above)}
\end{aligned}$$

However,

$$\begin{aligned}
\int_{-\infty}^\infty \frac{|\sigma + \imath\omega|^\mu}{|1 + \sigma + \imath\omega|^2} \mathrm{d}\omega &\le \int_{-\infty}^\infty \frac{\mathrm{d}\omega}{((1+\sigma)^2 + \omega^2)^{1-\mu/2}} \\
&= \frac{1}{(1+\sigma)^\varepsilon} \int_{-\infty}^\infty \frac{\mathrm{d}\omega}{(1+\omega^2)^{(\varepsilon+1)/2}} \\
&= \frac{1}{(1+\sigma)^\varepsilon} \sqrt{\pi} \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+1)/2)}.
\end{aligned}$$

Taking $\sigma = t^{-1}$, we can estimate

$$\|(a * g)(t)\| \leq C_\varepsilon(t) C_\mathrm{A}(t^{-1}) \int_0^\infty \|(\mathcal{P}_2 g)(\tau)\| \mathrm{d}\tau$$

with This is a bound similar to (3.7) with $\|\mathcal{P}_2 g\|$ integrated over $(0, \infty)$ instead of $(0, t)$. Let us fix $t > 0$ now and consider the function

$$p(\tau) := \begin{cases} g(\tau), & \tau \leq t, \\ e^{t-\tau}(g(t) + (\tau - t)(g(t) + \dot{g}(t)), & \tau > t. \end{cases}$$

Since p satisfies the same hypotheses as g, $a * p$ has the same properties as $a * g$. Also $p - g$ vanishes in $(-\infty, t)$ and, hence, by Proposition 3.2.1, the continuous function $a * (g - p)$ vanishes in $(-\infty, t)$ and therefore

$$\begin{aligned} \|(a * g)(t)\| = \|(a * p)(t)\| &\leq C_\varepsilon(t) C_\mathrm{A}(t^{-1}) \int_0^\infty \|(\mathcal{P}_2 p)(\tau)\| \mathrm{d}\tau \\ &= C_\varepsilon(t) C_\mathrm{A}(t^{-1}) \int_0^t \|(\mathcal{P}_2 g)(\tau)\| \mathrm{d}\tau, \end{aligned}$$

due to the fact that $\mathcal{P}_2 p = 0$ in (t, ∞). (To see this avoiding a lengthy computation, note that $p(\tau) = e^{-\tau}(e^t g(t) + (\tau - t)\frac{\mathrm{d}}{\mathrm{d}t}(e^{\cdot} g)(t))$ in (t, ∞).)
For $m \geq 1$, we consider $b \in \mathrm{TD}(\mathcal{B}(X, Y))$ such that $\mathcal{L}\{b\}(s) = s^{-m}\mathrm{A}(s)$ and note that $b * g = a * g^{(m)}$. Since b satisfies the hypotheses of the theorem in the case we have proved and $C_\mathrm{B} = C_\mathrm{A}$, the result follows. □

The only advantage of this improved estimate is that we can susbtitute expressions like $\mathcal{P}_3 g$ by $\mathcal{P}_2 \dot{g}$ and $\mathcal{P}_4 g$ by $\mathcal{P}_4 \ddot{g}$. (There are plenty of those.)