Retarded potentials and time domain boundary integral equations: a road-map

Francisco-Javier Sayas
University of Delaware

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Contents

Warning. These notes are under construction. (We apologize for any inconvenience!) Check out my web site (you just need to google my name) for the latest version. I will leave it in a prominent location for the time being.

These notes were first written written as support for a five hour lecture series *Retarded boundary integral equations and applications* as part of the closure workshop of the special semester on *Theoretical and numerical aspects of inverse problems and scattering theory.* The workshop took place in La Coruña, Spain, on July 4-8, 2011. The organizers of the event are warmly thanked for having thought of me for this occasion. The current version (March 19, 2013) has been expanded using recent results on time domain analysis.

In this notes I am trying to show the way through the relatively difficult theory of retarded layer potentials and integral operators for the acoustic wave equation in two and three dimensions. I will also introduce Convolution Quadrature techniques for the time discretization of potentials and integral equations and give a taste of their difficult but exciting theory and their huge not entirely explored potentialities. In a final chapter I will show some recent developments related to construction of integral absorbing boundary conditions for wave propagation problems. Part of the aim of these notes has been to set a clear path to learn the mathematical techniques for the use of time domain boundary integral equations. This is part of a joint effort with Antonio Laliena (Universidad de Zaragoza, Spain), the research group of Lehel Banjai at the Max-Planck-Institut in Leipzig (Germany) and my own group at the University of Delaware. Since this is just a working document, prepared for learning purposes, the tone will be somewhat colloquial. Apart from some more narrative sections (those with less mathematical rigor), everything else will be duly divided into paragraphs (plus propositions and their proofs) so that at each moment we know where we are.

The contents so far

1. Informal presentation of the retarded layer potentials. Derivation of the corresponding boundary integral calculus based on a few elements (the potentials and a uniqueness theorem). Use of layer potentials for scattering problems. This chapter is, again, informal.

2. Introduction of the basic tools for vector valued distributions and their Laplace transforms. (Not all the proofs will be given here but all steps will be duly sketched.) Distributional form of the problem of scattering by an obstacle. Laplace transform of the single layer potential and operator. Bounds depending on the Laplace transform parameter for all of them.

3. The formulas for (strong) inversion of the Laplace transform and the differentiation theorem are used to delimit a precise class of symbols (Laplace transforms) and their

time domain distributional counterparts. Convolution operators with this class of distributions is the setting for the remainder of the theory. We show how layer potentials, boundary integral operators and their inverses (whenever they exist) are in this class. A rigorous proof of Kirchhoff's formula (the integral representation theorem for causal acoustic waves) gives the necessary justification for the Calderón type calculus we had introduced in the first chapter. We finally have a look at how causality, finite speed of propagation and some kind of coercivity are hidden in the Laplace transform of the potentials and operators.

4. Of the two classes of Convolution Quadrature methods, I present here the one that is based on multistep methods. I present an almost finished portrait of the theory of these methods applied to the class of convolution operators and equations that was introduced in the previous chapter and detail the kind of results that are derived in the case of scattering by a sound soft obstacle.

5. We go back to the Single Layer retarded potential and go as far as we can with the Laplace domain techniques to prove estimates for the full discretization (Galerkin in space, Convolution Quadrature in time) of the model equation that solves the scattering problem by a sound-soft obstacle using an indirect formulation.

6. The following chapter is a simplified introduction to a class of abstract differential equations of the second order in Hilbert spaces. The hypotheses are much reduced with respect to what is common in the Hille-Yosida theory, but they will be those that we will meet later on. All results will be proved using quite rudimentary arguments of separation of variables space-time, related to the discrete spectrum of a given unbounded operator.

7. The techniques of the previous chapter are then used to prove again all the estimates for the single layer retarded potential and operator (as well as general Galerkin semidiscretization-in-space for the associated equation) using time domain estimates. We will develop a streamlined way of proving the time domain results, by working on a cut-off domain and identifying the resulting solution with the beginning of the evolution of the potential solution.

8. The following chapter repeats all Laplace and time domain argument on the double layer potential and its use for an indirect formulation of the scattering problem by a sound-hard obstacle. As the reader will easily realize at this time, the arguments end up being very similar in each particular situation and we will only have to take care of whatever is different in each concrete problem.

Although the theory of time domain boundary integral equations is far from finished (as its full potential in applications is only partially exploited), let me drop here some names of some of the originators of the current excitement in the area. This is a highly non-exhaustive list, so please, nobody take offence if their name does not appear here.

- The theory of time domain boundary integral equations (at least, the theory that we the numerical analysts use) stems from two papers by Alain Bamberger and Tuong Ha-Duong [2, 3] in 1986. Many other papers were published and even more theses were written (unfortunately much of this material was left unpublished and is now very difficult to locate) in the buoyant French numerical analysis school. The names of Jean-Claude Nédélec (at the Polytechnique) and Alain Bachelot (at the University of Bordeaux) are attached to quite a lot of these doctoral dissertations. Touffic Abboud and Isabelle Terrasse can be held responsible for the practical development of these methods, evolving in their (to the best of my knowledge) only commercial implementation. Interest in research aspects of this approach seems to be back: Abboud, Terrasse together with Patrick Joly (INRIA Rocquencourt, France) and Jerónimo Rodríguez (Santiago de Compostela, Spain) have recently developed one of the few sets of integral transparent boundary conditions [1].

- Convolution Quadrature originated as a completely independent tool for approximation of convolutions. It came to age very much at the same time as time domain integral equations, with two articles by Christian Lubich [14, 15] in 1988. This technique was first devised for problems with parabolic structure (exemplified in the operators having Laplace transforms defined on a sector instead of a half-plane). A second family of convolution quadrature methods, based on Runge-Kutta methods, originated in the joint work of Lubich with Alexander Ostermann [17]. Almost at the same time, Lubich applied his ideas to problems with hyperbolic structure, including the single layer potential for the three dimensional equation [16]. This was only natural, since CQ is based in Laplace transform methods and the theory of Bamberger & Ha-Duong is based on exactly the same principle. The theory of CQ based on RK schemes applied to hyperbolic problems was left unfinished and was only recently completed by Lubich in collaboration with Lehel Banjai and Jens Markus Melenk [4, 5].

- Not being as popular as their frequency domain cousins, time domain boundary integral equations have known a rich development in the engineering community. Methods based on Galerkin time domain discretization (which is the original point of view of the French school), applied to a vaste array of problems in electromagnetism, have been developed by the group of Eric Michielssen at the University of Michigan.

- The convolution quadrature point of view was initially not very well tended by the mathematical boundary integral community, but there was a strong development in the field of applications to elastodynamics, much of it led by the group of Martin Schanz at the Graz University of Technology (Austria). An early account of this development can be found in the monograph [21]. A more recent survey can be found in [6]. Applications to electromagnetism have been developed by the group of Daniel Weile and Peter Monk at the University of Delaware.

- Some papers by Stefan Suater (University of Zurich) with different collaborators, re-sparked the interest of numerical analysts in time domain integral equations, specially with a focus on convolution quadrature techniques. Lehel Banjai and his group are making rapid progress in this direction. Myself, working with my then graduate student Antonio Laliena, proved that the Laplace domain contained much more information than we had expected and that convolution quadrature techniques combined perfectly with space Galerkin discretization in many non-trivial situations of scattering of acoustic and elastic waves with penetrable obstacles, including non-homogeneous obstacles where numerical modeling is carried out with the finite element method [13]. We were happy to find a quite general (and I want to say innovative) approach that has since been applied to electromagnetism or more complicated elastodynamic problems.

- The full Galerkin approach is also being jointly developed by the groups of Ernst Stephan at the University of Hanover (Germany) and Matthias Maischak at Brunel University (England) with a current focus on acoustics. Several researchers in Italy (among them, Alessandra Aimi and Mauro Diligenti at Parma) are also readdressing the full Galerkin method for the acoustic equations.

Acknowledgements. As already mentioned, a great deal of what appears in this document is the result of continuous collaboration with Antonio Laliena, Víctor Domínguez, and Lehel Banjai. Some of my current students (Tonatiuh Sánchez-Vizuet and Tianyu Qiu) have had a careful look at several chapters of these notes, working out all the exercises and checking proofs. My current research is partially supported by the NSF (DMS ....). Three different meetings at the Oberwolfach Mathematical Institute, and in particular the talks I gave there, helped me in the search of a systematic approach to the development of this theory. I am deeply grateful to the organizers of those workshops (Martin Costabel and Ernst Stephan in the first one; Ralf Hiptmair, Roland Hoppe, Patrick Joly, and Ulrich Langer for the last two) for giving me the chance to enjoy the wonders of working, thinking and discussing in the middle of the Black Forest. Once again, these notes were triggered by an invitation to teach a summer course in Spain in 2011, and the organizers of that meeting can be blamed for starting these notes.

Chapter 1

The retarded layer potentials

In this chapter we are going to introduce the basic concepts of time domain acoustic layer potentials and how they can be used to represent the solutions of scattering problems. *All notions introduced in this chapter will be given at an intuitive level and with basically no formalization. The reader will have to wait for the next chapters to receive a precise sketch of the theory.*

1.1 Acoustic sources and dipoles

Let start this chapter by having a look at a spherical wave. We consider a function (a signal) $\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(t) = 0$ for all $t < 0$. A function of the time variable that vanishes for $t < 0$ will be always referred to as a **causal function**. We now choose $\mathbf{x}_0 \in \mathbb{R}^3$ and consider the function

$$u(\mathbf{x}, t) := \frac{\lambda(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|)}{4\pi|\mathbf{x} - \mathbf{x}_0|}. \tag{1.1}$$

A more or less boring computation shows that

$$c^{-2}\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \forall \mathbf{x} \neq \mathbf{x}_0 \quad \forall t > 0,$$

as long as $\lambda \in \mathcal{C}^2(\mathbb{R})$, where the Laplace operator Δ is taken in the space variables. (The result is actually true for less smooth λ, but we are not going to worry about regularity at this point.) It is interesting to notice the following facts.

- The function u moves on spherical surfaces. Actually,

  $$u(\mathbf{x}, t) = \frac{\lambda(t - c^{-1}r)}{4\pi r} \qquad |\mathbf{x} - \mathbf{x}_0| = r. \tag{1.2}$$

  This shows that points on a sphere centered at $\mathbf{x}_0$ perceive the same solution at the same time.

- The previous formula shows also that for a point at distance r of the point source, we need to wait $c^{-1}r$ time units to start perceiving any signal. Apart from this delay, the entire signal is received at speed c (and with a damping factor $4\pi r$). The signal goes through, exactly as emitted.

There are other kind of solutions of the wave equation that can be understood as traveling on spherical surfaces. If u is a sufficiently smooth solution of the wave equation, so are the three components of ∇u and therefore, so is $\nabla u \cdot \mathbf{d}$, where $\mathbf{d}$ is a fixed vector. With this idea, and starting in (1.1), we can create a new family of solutions to the wave equation:

$$\begin{aligned} u(\mathbf{x},t) &:= \nabla_{\mathbf{x}_0}\left(\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right)\cdot\mathbf{n}_0 \\ &= -\nabla_{\mathbf{x}}\left(\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right)\cdot\mathbf{n}_0 \\ &= \varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0}{4\pi|\mathbf{x}-\mathbf{x}_0|^3}+c^{-1}\dot\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0}{4\pi|\mathbf{x}-\mathbf{x}_0|^2} \end{aligned} \tag{1.3}$$

We will assume that $\mathbf{n}_0$ is a unit vector. This formula bears some similitude with (1.1). For instance, at time t, all points on the surface $|\mathbf{x}-\mathbf{x}_0| = r$ receive information from the signal $\varphi(t)$ proceeding from the same time $(t - c^{-1}r)$, although, the points do not only get the value of the signal $\varphi(t-c^{-1}r)$, but also its trend $\dot\varphi(t-c^{-1}r)$. The main difference between the wave (1.3) and the spherical wave (1.1) is directionality. While points seeing the source in the direction $\mathbf{n}_0$ get to perceive the signal, all points such that $(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0 = 0$ are in a deaf spot and miss the entire signal. Actually, if the angle between $\mathbf{x}-\mathbf{x}_0$ and $\mathbf{n}_0$ is θ, then

$$u(\mathbf{x},t) = \frac{1}{4\pi r}\Big(\frac{\varphi(t-c^{-1}r)}{r}+\frac{\dot\varphi(t-c^{-1}r)}{c}\Big)\cos\theta \tag{1.4}$$

The points $\mathbf{x}_0 \pm r\mathbf{n}_0$ (respective North and South pole of the sphere with axis $\mathbf{n}_0$, get the signal with the same amount of attenuation, but mirrored. The reader is encouraged to check the dimensions of all the elements in formulas (1.2) and (1.4) to recognize that the respective transmitted signals (λ and φ) have different dimensions. (One way to understand why is that there is differentiation in space in (1.3)-(1.4) which needs some kind of compensation.)

Another way of motivating the directional spherical wave (1.3) uses the physical idea of dipole. Take two source points

$$\mathbf{x}_0 \pm \tfrac{h}{2}\mathbf{n}_0,$$

separated a distance h in the direction $\mathbf{n}_0$. The upper point $\mathbf{x}_0 + \frac{h}{2}\mathbf{n}_0$ emits a signal $h^{-1}\varphi$ and simultaneously the point $\mathbf{x}_0 - \frac{h}{2}\mathbf{n}_0$ emits the signal $-h^{-1}\varphi$. The receiver gets to listen the signal

$$\frac{1}{h}\Big(\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0-\frac{h}{2}\mathbf{n}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0-\frac{h}{2}\mathbf{n}_0|}-\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0+\frac{h}{2}\mathbf{n}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0+\frac{h}{2}\mathbf{n}_0|}\Big)$$

which in the limit $h \to 0$ turns into (1.3).

1.2 Acoustic layer potentials

The single layer potential can be understood as the (continuous) superposition of spherical waves (1.1) being emitted from points on a surface Γ:

$$(\mathcal{S} * \lambda)(\mathbf{x},t) := \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}). \tag{1.5}$$

The causal signal $\lambda(t)$ has been substituted by a density distribution of causal signals $\lambda(\mathbf{y}, t)$, i.e., $\lambda : \Gamma \times \mathbb{R} \to \mathbb{R}$ such that $\lambda(\,\cdot\,, t) \equiv 0$ for $t < 0$. The convolution sign in the notation of this and the double layer potential is purely formal for the time being.

The reader who meets this kind of potential expression for the first time is encouraged to have a close look at the relatively bad aspect that it has: there is integration in the space variable $\mathbf{y}$ that somehow got its way into the time variable (through the delay). A particular set of densities is the addition of tensor products of functions of space and time

$$\lambda(\mathbf{y}, t) = \sum_{j=1}^{N} \Phi_j(\mathbf{y})\lambda_j(t),$$

producing simpler propagated signals

$$\sum_{j=1}^{N} \int_\Gamma \frac{\Phi_j(\mathbf{y})\lambda_j(t - c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} \mathrm{d}\Gamma(\mathbf{y}).$$

Simplifying even more, we can assume that the surface Γ has been subdivided into N panels $\{\Gamma_1, \ldots, \Gamma_N\}$ and Φ_j is just the characteristic function of the panel Γ_j. This is how the potential looks like now:

$$\sum_{j=1}^{N} \int_{\Gamma_j} \frac{\lambda_j(t - c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} \mathrm{d}\Gamma(\mathbf{y}).$$

In any of the above expressions, it is easy to check that if a point is at a distance r of Γ, it will take $T = c^{-1}r$ time units for the signal to reach the point. Apart from very simple configurations, different points $\mathbf{x}$ will perceive different outputs, since the balance of distances $|\mathbf{x}-\mathbf{y}|$ with the spacial distribution of the density is going to differ depending on the point of view.

Another class of signals we can plug into the potential expression are time-harmonic signals. A non-causal time harmonic signal emitted from Γ would be

$$\mathrm{Re}\,(\lambda(\mathbf{y})e^{-\imath\omega t}) \qquad \lambda : \Gamma \to \mathbb{C},$$

which is heard as a time harmonic signal

$$\mathrm{Re}\Big(e^{-\imath\omega t} \underbrace{\int_\Gamma \frac{e^{\imath\omega c^{-1}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\lambda(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})}\Big).$$

The underbraced expression can be recognized as a single layer potential associated to the Helmholtz equation $\Delta + k^2$, ($k = \omega/c$ is the wave number) which is the equation satisfied by the spacial part of a time harmonic solution to the wave equation.

A double layer potential can be defined with the same idea of superposition. The directionality at the point $\mathbf{y} \in \Gamma$ its given by the unit normal vector $\boldsymbol{\nu}(\mathbf{y})$:

$$\begin{aligned}(\mathcal{D} * \varphi)(\mathbf{x}, t) \;:=&\; \int_\Gamma \nabla_\mathbf{y}\left(\frac{\varphi(\mathbf{z}, t - c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\Big|_{\mathbf{z}=\mathbf{y}} \cdot \boldsymbol{\nu}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ =&\; \int_\Gamma \frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\big(\varphi(\mathbf{y}, t - c^{-1}|\mathbf{x}-\mathbf{y}|) + c^{-1}|\mathbf{x}-\mathbf{y}|\dot\varphi(\mathbf{y}, t - c^{-1}|\mathbf{x}-\mathbf{y}|)\big)\mathrm{d}\Gamma(\mathbf{y}).\end{aligned}$$

Obviously, for this expression to make sense we need an orientable surface with a well defined normal vector field (almost everywhere, so polyhedra are not a problem).

1.3 Jump relations

Let us try to see some properties of the possible limits of the layer potentials when we get close to the surface.

Continuity of the single layer potentials. A possible way to study the single layer potential is by studying functions of the form

$$w(\mathbf{x},\widehat{\mathbf{x}},t) := \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\widehat{\mathbf{x}}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}), \tag{1.6}$$

since $(\mathcal{S}*\lambda)(\mathbf{x},t) = w(\mathbf{x},\mathbf{x},t)$. Let $\mathbf{z}\in\Gamma$. We first take the limit $\widehat{\mathbf{x}}\to\mathbf{z}$ in (1.6) and we obtain (formally at least)

$$\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{z}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}). \tag{1.7}$$

In a second step, we recognize in (1.7) the form of a Coulomb potential (the single layer potential for the Laplacian), which is continuous across Γ. This means that

$$\lim_{\mathbf{x}\to\mathbf{z}\in\Gamma}(\mathcal{S}*\lambda)(\mathbf{x},t) = \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{z}-\mathbf{y}|)}{4\pi|\mathbf{z}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}) =: (\mathcal{V}*\lambda)(\mathbf{z},t).$$

Discontinuity of the normal derivative of the single layer potential. We next look at directional derivatives of $\mathcal{S}*\lambda$. Let $\boldsymbol{\nu}=\boldsymbol{\nu}(\mathbf{z})$ with $\mathbf{z}\in\Gamma$. Then:

$$\begin{aligned}\big(\nabla_\mathbf{x}(\mathcal{S}*\lambda)\cdot\boldsymbol{\nu}\big)(\mathbf{x},t) \quad &= \quad -c^{-1}\int_\Gamma \frac{\dot\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y})\\ &\qquad -\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y})\\ &=: \quad a(\mathbf{x},t)+b(\mathbf{x},t)\end{aligned}$$

With arguments similar to those we used in the continuity analysis of $\mathcal{S}*\lambda$, we can prove that a is continuous across Γ. We are now going to give a simplified argument demonstrating that

$$b(\mathbf{z}-\varepsilon\boldsymbol{\nu}(\mathbf{z}),t) - b(\mathbf{z}+\varepsilon\boldsymbol{\nu}(\mathbf{z}),t) \stackrel{\varepsilon\to 0}{\longrightarrow} \lambda(\mathbf{z},t),$$

which is equivalent to showing that the jump of the normal derivative of $\mathcal{S}*\lambda$ across Γ is λ. Note that when $\mathbf{x}\to\mathbf{z}\in\Gamma$, only a neighborhood of $\mathbf{z}$ in Γ is relevant from the point of view of creating a discontinuity in the integral:

$$b(\mathbf{x},t) = -\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y}).$$

To further simplify the exposition, let us assume that Γ is a flat surface around $\mathbf{z}$. After translation, rotation and localization, we can assume that

$$\mathbf{z}=\mathbf{0} \qquad \boldsymbol{\nu}=(0,0,1) \qquad \Gamma=\{(\mathbf{y},0)\,:\,\mathbf{y}\in\mathbb{R}^2\,|\mathbf{y}|<R\}=B(0,R)\times\{0\}.$$

If $\mathbf{x}=\mathbf{z}\pm\varepsilon\boldsymbol{\nu}(\mathbf{z})=\pm\varepsilon(0,0,1)$, then

$$\begin{aligned}
b(\mathbf{0}-\varepsilon\boldsymbol{\nu},t)-b(\mathbf{0}+\varepsilon\boldsymbol{\nu},t) &= \varepsilon\int_{B(0,R))}\frac{\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|)}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}\\
&= \lambda(\mathbf{0},t)\int_{B(0,R)}\frac{\varepsilon}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}\\
&\quad +\varepsilon\int_{B(0,R)}\frac{\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|)-\lambda(\mathbf{0},t)}{4\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}. \qquad (1.8)
\end{aligned}$$

Note that

$$\int_{B(0,R)}\frac{\varepsilon}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}=\int_0^R\frac{\varepsilon\, r}{\sqrt{(r^2+\varepsilon^2)^3}}\mathrm{d}r=1-\frac{\varepsilon}{\sqrt{R^2+\varepsilon^2}}\xrightarrow{\varepsilon\to 0}1. \qquad (1.9)$$

On the other hand, for smooth λ

$$|\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|)-\lambda(\mathbf{0},t)|\le C_1|\mathbf{y}|+C_2\varepsilon, \qquad (1.10)$$

$$\int_{B(0,R)}\frac{\varepsilon^2}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}\xrightarrow{\varepsilon\to 0}0, \qquad (1.11)$$

and

$$\begin{aligned}
\int_{B(0,R)}\frac{\varepsilon|\mathbf{y}|}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} &= \int_0^R\frac{\varepsilon r^2}{\sqrt{(r^2+\varepsilon^2)^3}}\mathrm{d}r\\
&= \varepsilon\log(\sqrt{R^2+\varepsilon^2}+R)-\frac{\varepsilon R}{\sqrt{R^2+\varepsilon^2}}-\varepsilon\log\varepsilon\xrightarrow{\varepsilon\to 0}0. \qquad (1.12)
\end{aligned}$$

Using (1.9), (1.10), (1.11) and (1.12) in (1.8), the result follows. The case of curved boundaries is very similar.

Discontinuity of the double layer potential. The expression for the double layer potential

$$\begin{aligned}
(\mathcal{D}*\varphi)(\mathbf{x},t) &= c^{-1}\int_\Gamma\frac{\dot\varphi(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y})\\
&\quad +\int_\Gamma\frac{\varphi(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y})
\end{aligned}$$

definitely resembles that of the directional derivative of the single layer potential. We can recognize two terms: the first one is continuous and in the second one, we can use the same argument (as in exactly the same argument) to prove that for every $\mathbf{z}\in\Gamma$ such that Γ is flat around $\mathbf{z}$

$$(\mathcal{D}*\varphi)(\mathbf{z}+h\boldsymbol{\nu}(\mathbf{z}),t)-(\mathcal{D}*\varphi)(\mathbf{z}-h\boldsymbol{\nu}(\mathbf{z}),t)\xrightarrow{\varepsilon\to 0}\varphi(\mathbf{z},t).$$

Note that the sign of the jump is the opposite to the one of the normal derivative of $\mathcal{S}*\lambda$.

Continuity of the normal derivative of the double layer potential. Assuming more regularity for the density φ, it is possible to show that the normal derivative of $\mathcal{D} * \varphi$ is continuous across smooth points of Γ. The proof is more involved (tangential integration by parts is involved and finite part integrals make their appearance) and requires a certain amount of patience. Because we will take a different point of view, using Laplace transform techniques and basing our results on well established properties of layer potentials for elliptic problems, we will just accept this result for the moment being.

1.4 A Calderón type calculus

The structure of the boundary integral calculus for the wave equation is very similar to that of elliptic operators, so those accustomed to the many formulas (Green representation theorem, boundary integral identities, Calderón projector, etc) of the boundary integral calculus will recognize here exactly the same basic structure. The main difference is at the analytic level: spaces are much less clear and the theory requires quite some effort to be developed. The boundary integral calculus can be derived in several ways. My favorite is the following. It develops from three concepts:

- a uniqueness theorem for transmission problems,
- a concept of single layer operator,
- a concept of double layer operator.

(The three concepts can be grouped in one: an existence and uniqueness theorem for transmission problems.) Once these elements have been established, the representation theorem (Green's Theorem for steady state problems, Kirchhoff's formula for waves) is a direct consequence of these elements. The boundary integral operators are the averages of the Cauchy data of layer operators and they yield a collection of integral identities satisfied by interior and exterior solutions.

We are going to informally expose this theory. We will need Chapters 2 and 3 to develop a rigorous theory for the main building blocks. The geometric layout is composed of a bounded domain Ω^-, with Lipschizt boundary Γ and exterior $\Omega^+ := \mathbb{R}^d \setminus \Gamma$ (that is supposed to be connected). The restriction (trace) of a function u to the boundary Γ from the interior and exterior of Γ will be denoted $\gamma^- u$ and $\gamma^+ u$ respectively. The normal derivative (with the normal vector pointing outwards) from inside and outside are $\partial_\nu^- u$ and $\partial_\nu^+ u$. Jumps of these two quantities across the interface Γ are denoted

$$[\![\gamma u]\!] := \gamma^- u - \gamma^+ u, \qquad [\![\partial_\nu u]\!] := \partial_\nu^- u - \partial_\nu^+ u.$$

Averages are denoted with double curly brackets

$$\{\!\{\gamma u\}\!\} := \tfrac{1}{2}(\gamma^- u + \gamma^+ u), \qquad \{\!\{\partial_\nu u\}\!\} := \tfrac{1}{2}(\partial_\nu^- u + \partial_\nu^+ u).$$

In the background of this theory there is a class of functions $u(\mathbf{x}, t)$ for which we can take second time derivatives, spacial Laplacian, traces and normal derivatives on the boundary and initial values at $t = 0$. For the moment being let us refer to these functions as admissible functions.

The uniqueness result. The first key result is uniqueness result for a kind of transmission problem of the wave equation. It can be informally stated as follows: if an admissible function u satisfies

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty), \\ [\![\gamma u]\!] &= 0 && \text{on } \Gamma \times (0,\infty), \\ [\![\partial_\nu u]\!] &= 0 && \text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \end{aligned}$$

then u is necessarily zero.

Those used to frequency domain problems will be wondering where the radiation condition is. This can be dealt with in several ways, demanding finite energy for each time, asking for bounded spacial support for each time, etc. At this level, we assume that this is part of the class of functions where we express uniqueness. When we develop the correct theoretical frame, radiation will be part of causality and will not have to be expressed as a separate condition.

A single layer potential. For a function $\lambda : \Gamma \times (0,\infty) \to \mathbb{R}$ in a certain class of functions, there exists and admissible function $u := \mathcal{S} * \lambda$ such that

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty), \\ [\![\gamma u]\!] &= 0 && \text{on } \Gamma \times (0,\infty), \\ [\![\partial_\nu u]\!] &= \lambda && \text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma. \end{aligned}$$

This is, obviously, the unique solution of this problem.

A double layer potential. For a function $\varphi : \Gamma \times (0,\infty) \to \mathbb{R}$ in certain class, there exists and admissible function $u := \mathcal{D} * \varphi$ such that

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty), \\ [\![\gamma u]\!] &= -\varphi && \text{on } \Gamma \times (0,\infty), \\ [\![\partial_\nu u]\!] &= 0 && \text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma. \end{aligned}$$

There is an inherent compatibility condition between the three classes of functions. It can be expressed as follows: given u in the class of pressure (wave) fields, the quantities $\lambda := [\![\partial_\nu u]\!]$ and $\varphi := [\![\gamma u]\!]$ can be used as respective inputs of the single and double layer potentials.

First consequence: Kirchhoff's formula. If u is a solution of the wave equation around Γ

$$\begin{aligned} c^{-2}u_{tt} = \Delta u \qquad & \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) = 0 \qquad & \text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) = 0 \qquad & \text{in } \mathbb{R}^d \setminus \Gamma, \end{aligned}$$

then

$$u = \mathcal{S} * [\![\partial_\nu u]\!] - \mathcal{D} * [\![\gamma u]\!]. \tag{1.13}$$

This is a direct consequence of the definitions of layer potentials and the uniqueness theorem for transmission problems.

New definitions: boundary integral operators. The properties of potentials

$$[\![\gamma(\mathcal{S} * \lambda)]\!] = 0 \qquad [\![\partial_\nu(\mathcal{D} * \varphi)]\!] = 0 \tag{1.14}$$

allow us to define the following four operators:

$$\begin{aligned} \mathcal{V} * \lambda &:= \{\!\{\gamma(\mathcal{S} * \lambda)\}\!\} = \gamma^-(\mathcal{S} * \lambda) = \gamma^+(\mathcal{S} * \lambda), \\ \mathcal{K}^t * \lambda &:= \{\!\{\partial_\nu(\mathcal{S} * \lambda)\}\!\}, \\ \mathcal{K} * \varphi &:= \{\!\{\gamma(\mathcal{D} * \varphi)\}\!\}, \\ \mathcal{W} * \varphi &:= -\{\!\{\partial_\nu(\mathcal{D} * \varphi)\}\!\} = -\partial_\nu^-(\mathcal{D} * \varphi) = -\partial_\nu^+(\mathcal{D} * \varphi). \end{aligned}$$

Since

$$[\![\partial_\nu(\mathcal{S} * \lambda)]\!] = \lambda \qquad [\![\gamma(\mathcal{D} * \varphi)]\!] = 0,$$

the definitions imply that

$$\partial_\nu^\pm(\mathcal{S} * \lambda) = \mp\tfrac{1}{2}\lambda + \mathcal{K}^t * \lambda \qquad \gamma^\pm(\mathcal{D} * \varphi) = \pm\tfrac{1}{2}\varphi + \mathcal{K} * \varphi. \tag{1.15}$$

The collection of all these formulas is often referred to as **the jump relations** of potentials.

Boundary integral identities. Starting at the representation theorem (Kirchhoff's formula)

$$u = \mathcal{S} * [\![\partial_\nu u]\!] - \mathcal{D} * [\![\gamma u]\!]$$

and using the jump relations, we can write, for instance,

$$\begin{bmatrix} \{\!\{\gamma u\}\!\} \\ \{\!\{\partial_\nu u\}\!\} \end{bmatrix} = \begin{bmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}^t \end{bmatrix} * \begin{bmatrix} [\![\gamma u]\!] \\ [\![\partial_\nu u]\!] \end{bmatrix}. \tag{1.16}$$

Exterior solutions: direct method. The previous presentation was carried out for solutions of transmission problems. The reader might wonder, what to do when we only have an exterior solution at our disposal, i.e., a solution of

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u && \text{in } \Omega^+ \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma. \end{aligned}$$

The simplest thing to do is to consider that $u \equiv 0$ in $\Omega^- \times (0,\infty)$ naturally completes the exterior solution. Then

$$[\![\gamma u]\!] = -\gamma^+ u, \quad \{\!\{\gamma u\}\!\} = \tfrac{1}{2}\gamma^+ u, \quad [\![\partial_\nu u]\!] = -\partial_\nu^+ u, \quad \{\!\{\partial_\nu u\}\!\} = \tfrac{1}{2}\partial_\nu^+ u.$$

Therefore, Kirchhoff's formula (1.13) for this u is reexpressed as

$$u = \mathcal{D} * \gamma^+ u - \mathcal{S} * \partial_\nu^+ u, \tag{1.17}$$

while the integral identities (1.16) become

$$\frac{1}{2}\begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \begin{bmatrix} \mathcal{K} & -\mathcal{V} \\ -\mathcal{W} & -\mathcal{K}^t \end{bmatrix} * \begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} \tag{1.18}$$

or equivalently

$$\begin{bmatrix} \frac{\delta_0}{2} - \mathcal{K} & \mathcal{V} \\ \mathcal{W} & \frac{\delta_0}{2} + \mathcal{K}^t \end{bmatrix} * \begin{bmatrix} \gamma^+ u \\ \partial_\nu^+ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.19}$$

In this last expression we have used the formal notation $\delta_0 * \xi = \xi$ (convolution with the Dirac delta is the identity) in order to keep convolutional notation for the entire matrix of operators. Some people prefer reordering the columns in the above expression to get

$$\begin{bmatrix} \mathcal{V} & \frac{\delta_0}{2} - \mathcal{K} \\ \frac{\delta_0}{2} + \mathcal{K}^t & \mathcal{W} \end{bmatrix} * \begin{bmatrix} \partial_\nu^+ u \\ \gamma^+ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

emphasizing some kind of symmetry of the system that is not clear at the present moment.

It is interesting to remark that Kirchhoff's formula for the exterior solution is also representing the zero solution inside (that is how we got to it) and therefore

$$\mathcal{D} * \gamma^+ u - \mathcal{S} * \partial_\nu^+ u \equiv 0 \qquad \text{in } \Omega^-.$$

1.5 Scattering problems

The geometric layout of this section will be exactly the same as the one of the previous section: a bounded domain Ω^- with Lipschitz boundary Γ (this is the scatterer) and connected exterior Ω^+.

Incident waves. In a scattering problem there is something called an incident wave that is supposed to be known. Possible incident waves are spherical waves

$$u^{\mathrm{inc}}(\mathbf{x},t)=\frac{\lambda(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0|},$$

emitted from a point source $\mathbf{x}_0\notin\overline{\Omega^-}$ and with causal signal λ. Their directional counterparts

$$u^{\mathrm{inc}}=\nabla_{\mathbf{x}_0}\left(\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right)\cdot\mathbf{n}_0$$

are also valid incident waves. Superposition of several of the above are also acceptable incident waves. These waves are compactly supported at all times. Plane waves make for good non-compactly supported incident waves

$$u^{\mathrm{inc}}(\mathbf{x},t)=\lambda(c^{-1}\mathbf{x}\cdot\mathbf{d}-t)\qquad|\mathbf{d}|=1.$$

For the placement of the obstacle to be feasible we need that at time zero the support of $u^{\mathrm{inc}}(\,\cdot\,,0)$ and $u^{\mathrm{inc}}_t(\,\cdot\,,0)$ does not intersect $\overline{\Omega^-}$. Apart from compactness of their support, the nature of spherical and plane waves is very different, since, when seen in free space $\mathbb{R}^d$, the first ones are solutions of a non-homogeneous wave equation (there is a singular source placed at $\mathbf{x}_0$) but have vanishing initial condition, while plane waves are unforced solutions of the wave equation corresponding to a particular set of non-compactly supported initial conditions.

In all cases, it is customary to consider that an incident wave is a solution of a wave propagation problem in free space:

$$\begin{aligned}c^{-2}u^{\mathrm{inc}}_{tt}&=\Delta u^{\mathrm{inc}}+f&&\text{in }\mathbb{R}^d\times(0,\infty),\\u^{\mathrm{inc}}(\,\cdot\,,0)&=u_0&&\text{in }\mathbb{R}^d,\\u^{\mathrm{inc}}_t(\,\cdot\,,0)&=v_0&&\text{in }\mathbb{R}^d.\end{aligned}$$

Physical placement of the obstacle at time $t=0$ requires that the support of u_0 and v_0 does not intersect $\overline{\Omega^-}$. Source terms at any given time $f(\,\cdot\,,t)$, should be set apart from the scatterer as well.

Scattering by obstacles. The influence of the simplest scatterers on the wave field is modeled by a boundary condition. For instance, a sound-soft obstacle induces a Dirichlet boundary condition:

$$\begin{aligned}c^{-2}u^{\mathrm{tot}}_{tt}&=\Delta u^{\mathrm{tot}}+f&&\text{in }\Omega^+\times(0,\infty),\\\gamma u^{\mathrm{tot}}&=0&&\text{on }\Gamma\times(0,\infty),\\u^{\mathrm{tot}}(\,\cdot\,,0)&=u_0&&\text{in }\Omega^+,\\u^{\mathrm{tot}}_t(\,\cdot\,,0)&=\dot u_0&&\text{in }\Omega^+.\end{aligned}$$

The source term and initial conditions are the same as for the incident wave. A sound-hard obstacle is modeled with a Neumann boundary condition instead

$$\partial_\nu u^{\mathrm{tot}}=0\qquad\text{on }\Gamma\times(0,\infty).$$

Simple absorbing boundary conditions are modeled with the dynamic boundary conditions

$$\partial_\nu u^{\rm tot} - \alpha c^{-1} \gamma u_t = 0 \qquad \text{on } \Gamma \times (0,\infty),$$

where $\alpha : \Gamma \to [0,\infty)$ plays the role of a surface impedance function.

The scattered wave. For practical (computational) purposes, the decomposition

$$u^{\rm tot} = u^{\rm inc} + u^{\rm scat}$$

is often used. Note that this is a non-physical decomposition. The scattered wave field is just the difference between what is happening ($u^{\rm tot}$) and what would happen if the obstacle were not present ($u^{\rm inc}$). While the scattered wave will respect some basic causality principles and will travel at the correct speed, it will do so without noticing the obstacle, because it has to compensate for the fact that the incident wave does not perceive the obstacle while, obviously, the total field does. Our future unknown will be the scattered wave field $u := u^{\rm scat} = u^{\rm tot} - u^{\rm inc}$ that satisfies a homogeneous equation, with homogeneous initial conditions but non-vanishing boundary conditions:

$$\begin{aligned} c^{-2} u_{tt} &= \Delta u && \text{in } \Omega^+ \times (0,\infty), \\ \mathrm{BC}(u) &= -\mathrm{BC}(u^{\rm inc}) && \text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \Omega^+, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \Omega^+. \end{aligned}$$

Other possible scattering problems include penetrable scatterers with different material properties. Combined methods, using integral equations for the exterior scattered field and volume formulations for the interior of the scatterer are available. Mathematically speaking, they do not differ in essence to the kind of problems we meet when building transparent boundary conditions for wave propagation problems (Chapter 5).

1.6 Integral formulations in scattering

Consider again the problem of scattering by a sound-soft obstacle:

$$\begin{aligned} c^{-2} u_{tt} &= \Delta u && \text{in } \Omega^+ \times (0,\infty), \\ \gamma u &= g && \text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \Omega^+, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \Omega^+. \end{aligned}$$

The incident wave is only perceived as a boundary condition

$$g := -\gamma u^{\rm inc} : \Gamma \times (0,\infty) \to \mathbb{R}.$$

Indirect formulation. Since single-layer potentials are solutions of the wave equation with homogeneous initial conditions, we are allowed to propose

$$u := \mathcal{S} * \lambda \tag{1.20}$$

as a possible solution of the scattering problem, for a causal density to be determined. Recalling that we have denoted $\mathcal{V} * \lambda := \gamma(\mathcal{S} * \lambda)$, it follows that in order for $u = \mathcal{S} * \lambda$ to be a solution, we need

$$\mathcal{V} * \lambda = g. \tag{1.21}$$

In due time, we will see that (1.21) is a uniquely solvable equation. With it we will determine the density λ and the representation formula (1.20) will deliver the solution to the scattering problem. Because single-layer potentials are continuous across the interface Γ, we are naturally attaching a continuous extension of the solution to the interior of the scatterer, so with (1.20)-(1.21) we are actually solving

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u & &\text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty), \\ \gamma^{\pm} u &= g & &\text{on } \Gamma \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 & &\text{in } \mathbb{R}^d \setminus \Gamma, \\ u_t(\,\cdot\,,0) &= 0 & &\text{in } \mathbb{R}^d \setminus \Gamma. \end{aligned}$$

These are two separate wave equations (one in Ω^+ and another one in Ω^-), that have no connection apart from sharing the same boundary condition (this is not a transmission problem). When $g = -\gamma u^{\text{inc}}$ it follows that the interior solution is just $-u^{\text{inc}}$.

Direct formulation. Another approach consists of taking Kirchhoff's formula as the integral representation formula. Therefore, we have (recall (1.17))

$$u = \mathcal{D} * \gamma^+ u - \mathcal{S} * \partial_\nu^+ u.$$

Setting $\lambda := \partial_\nu^+ u$ to be our boundary unknown and substituting the boundary condition, it follows that

$$u = \mathcal{D} * g - \mathcal{S} * \lambda. \tag{1.22}$$

The first of the integral identities (1.18) or (1.19)

$$\mathcal{V} * \lambda = -\tfrac{1}{2} g + \mathcal{K} * g \tag{1.23}$$

can be used as an integral equation in order to try to find λ and obtain the scattered wavefield with it. While the pair of integral equation and integral representation for the indirect formulation (1.20)-(1.21) is definitely simpler, the pair (1.22)-(1.23) has the advantage of computing a physical quantity on the boundary. The practical choice in frequency domain problems is almost invariably for the direct method, but there is much more to it (for complicated reasons we will not deal with) in the time domain. There is also the question of whether we should strive for formulations where the boundary unknown appears under the action of an operator of the second kind (identity plus integral operator). I will not deal with this question either, since the choices in the time domain are still quite open.

1.7 Galerkin semidiscretization

We are going to do some work on the equation

$$\mathcal{V} * \lambda = g \tag{1.24}$$

that appeared in Section 1.6. In strong integral form, this is

$$\int_\Gamma \frac{\lambda(\mathbf{y}, t - c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} \mathrm{d}\Gamma(\mathbf{y}) = g(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma \qquad t > 0. \tag{1.25}$$

The unknown is a causal function $\lambda : \Gamma \times \mathbb{R} \to \mathbb{R}$. For (semi)discretization let us choose a finite dimensional space $X^h \subset L^\infty(\Gamma)$ with basis $\{\Phi_1, \ldots, \Phi_N\}$ and let us look for

$$\lambda^h := \sum_{j=1}^N \lambda_j(t) \Phi_j(\mathbf{x}), \tag{1.26}$$

where the unknowns are N causal functions $\lambda_j : \mathbb{R} \to \mathbb{R}$ of the time variable. Because we are drastically reducing the set of possible densities we cannot expect that the integral equation (1.24) stands any chance of having a solution of the form (1.26). Instead, we will look for some averages (moments) of left and right hand side to coincide, namely,

$$\int_\Gamma \Phi_i(\mathbf{x}) \big(\mathcal{V} * \lambda^h\big)(\mathbf{x}, t) \mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma \Phi_i(\mathbf{x})\, g(\mathbf{x}, t) \mathrm{d}\Gamma(\mathbf{x}) \qquad i = 1, \ldots, N, \qquad t > 0.$$

This is equivalent to the following system

$$\sum_{j=1}^N \int_\Gamma \int_\Gamma \frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \lambda_j(t - c^{-1}|\mathbf{x}-\mathbf{y}|)\, \mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma \Phi_i(\mathbf{x})\, g(\mathbf{x}, t) \mathrm{d}\Gamma(\mathbf{x}) \tag{1.27}$$

$$i = 1, \ldots, N, \qquad t > 0.$$

This is a system of delay equations with delays integrated over distances between points. In Chapter 4 we will see that this system can be considered as a system of convolution equations that can be discretized with the convolution quadrature method.

A second option consists of using a time-Galerkin discretization of the system above. This is equivalent to decomposing each λ_j as a linear combination of functions and testing the equations above in the following way: we multiply them **by the derivative (!!) of the shape functions** and possibly some fixed time-dependent weight, and we then integrate in the time variable.The inherent coercivity principle is explained in Section 3.7. To explain this more clearly, let us assume that we have a time-grid with constant time-step

$$0 = t_0 < t_1 < t_2 < \ldots < t_n < \ldots \qquad t_n = \kappa\, n,$$

and that we use piecewise constant functions in time. This corresponds to choosing the basis of characteristic functions

$$\chi_{(t_{n-1}, t_n)}(t) \qquad n \geq 1.$$

The fully discrete unknown has now the form

$$\sum_{j=1}^{N}\sum_{m=1}^{\infty}\lambda_j^m\,\chi_{(t_{m-1},t_m)}(t)\Phi_j(\mathbf{x}),\qquad \lambda_j^m\in\mathbb{R}.$$

Each of the equations (1.27) is multiplied by $\dot\chi_{(t_{n-1},t_n)}$ and integrated from 0 to ∞. Since $\dot\chi_{(t_{n-1},t_n)}=-\delta_{t_n}+\delta_{t_{n-1}}$, this is equivalent to setting up a system

$$\sum_{j=1}^{N}\sum_{m=1}^{n}\lambda_j^m(-A_{ij}^{n-m}+A_{ij}^{n-1-m})=-g_i^n+g_i^{n-1} \tag{1.28}$$

with

$$\begin{aligned}
A_{ij}^{n-m} &= \int_\Gamma\int_\Gamma\frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\chi_{(t_{m-1},t_m)}(t_n-c^{-1}|\mathbf{x}-\mathbf{y}|)\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y})\\
&= \int_\Gamma\int_\Gamma\frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\chi_{(t_n-t_m,t_n-t_{m-1})}(c^{-1}|\mathbf{x}-\mathbf{y}|)\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y})\\
&= \int_\Gamma\int_\Gamma\frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\chi_{(t_{n-m},t_{n-m+1})}(c^{-1}|\mathbf{x}-\mathbf{y}|)\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y})\\
&= \iint_{\Gamma_c^{n-m}}\frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}),
\end{aligned}$$

where

$$\Gamma_c^\ell:=\{(\mathbf{x},\mathbf{y})\in\Gamma\times\Gamma\,:\,ct_\ell\le|\mathbf{x}-\mathbf{y}|\le c\,t_{\ell+1}\},\qquad \ell\ge 0,$$

and

$$g_i^n=\int_\Gamma\Phi_i(\mathbf{x})g(\mathbf{x},t_n)\mathrm{d}\Gamma(\mathbf{x})$$

The system (1.28) has a block triangular structure with some Toeplitz character to it. (This character, as well as the convolutional structure in discrete time is heavily dependent on us having chosen a constant time-step.) We can start by solving for $n=1$

$$\sum_{j=1}^{N}A_{ij}^0\lambda_j^1=g_i^1\qquad i=1,\ldots,N.$$

Next we solve for $n=2$

$$\sum_{j=1}^{N}A_{ij}^0\lambda_j^2+\sum_{j=1}^{N}(A_{ij}^1-A_{ij}^0)\lambda_j^1=g_i^2-g_i^1,$$

for $n=3$,

$$\sum_{j=1}^{N}A_{ij}^0\lambda_j^3+\sum_{j=1}^{N}(A_{ij}^1-A_{ij}^0)\lambda_j^2+\sum_{j=1}^{N}(A_{ij}^2-A_{ij}^1)\lambda_j^0=g_i^3-g_i^2,$$

and so on. Each time-step requires the solution of a linear system with the same matrix. (If piecewise polynomial discontinuous functions are used for space discretization, the matrix A^0 is actually very sparse.) The sets Γ^ℓ are empty for large enough ℓ (depending on the size of Γ, the speed of waves c and the time step κ) and therefore the system (1.28) has a finite tail.

The main difficulty of this full Galerkin approach is related to the precise computation of the matrices A^n. Thinking of a piecewise constant approximation in the space variable, we need to compute integrals

$$\iint_{\Gamma^\ell_{i,j}} \frac{1}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) \qquad \Gamma^\ell_{i,j} := \{(\mathbf{x},\mathbf{y})\,:\,\mathbf{x}\in\Gamma_i,\,\mathbf{y}\in\Gamma_j,\,ct_\ell \le |\mathbf{x}-\mathbf{y}| \le ct_{\ell+1}\},$$

where a weakly singular integrand is integrated on domains with quite exotic shape. This is not easily done and it has been frequently reported that the method is extremely sensitive to small errors (and mistakes) in computation of these integrals.

1.8 Two dimensional waves

Single layer potential. The layer potentials in two dimensional are considerably more complicated. The single layer potential admits the expression:

$$\begin{aligned}(\mathcal{S}*\lambda)(\mathbf{x},t) \;&:=\; \frac{1}{2\pi}\int_\Gamma\int_0^{t-c^{-1}|\mathbf{x}-\mathbf{y}|} \frac{\lambda(\mathbf{y},\tau)}{\sqrt{(t-\tau)^2 - c^{-2}|\mathbf{x}-\mathbf{y}|^2}}\,\mathrm{d}\Gamma(\mathbf{y})\,\mathrm{d}\tau \\ &=\; \int_\Gamma\int_0^t G_2(t-\tau,|\mathbf{x}-\mathbf{y}|)\,\lambda(\mathbf{y},\tau)\,\mathrm{d}\Gamma(\mathbf{y})\mathrm{d}\tau\end{aligned}$$

where

$$G_2(t,r) := \frac{H(t-c^{-1}r)}{2\pi\sqrt{t^2 - c^{-2}r^2}}$$

and H is the Heaviside function. Unlike in the three dimensional case, there is a space-and-time integral in the definition, since the two dimensional wave kernel is a function. Compare this with the three dimensional kernel, that can be formally expressed as

$$G_3(t,r) := \frac{\delta(t-c^{-1}r)}{4\pi r}.$$

The Dirac delta distribution creates the pure delay (no integration over the past) of this kernel.

Why? There is a simple argument that shows how the two dimensional potential comes up. Assume that the causal signal $\lambda(t)$ is simultaneously emitted by all points in the z axis. A point $(x,y,0) \equiv (r\cos\theta, r\sin\theta, 0)$ then receives the signal

$$\int_{-\infty}^{\infty} \frac{\lambda(t-c^{-1}\sqrt{r^2+z^2})}{4\pi\sqrt{r^2+z^2}}\mathrm{d}z = 2\int_0^\infty \frac{\lambda(t-c^{-1}\sqrt{r^2+z^2})}{4\pi\sqrt{r^2+z^2}}\mathrm{d}z.$$

The change of variables (changing the third dimension z to a fictitious time τ)

$$(0,\infty) \ni z \longleftrightarrow \tau = t - c^{-1}\sqrt{r^2+z^2} \in (-\infty, t-c^{-1}r),$$

$$\frac{\mathrm{d}z}{\sqrt{z^2+r^2}} = \frac{\mathrm{d}\tau}{zc^{-1}} = \frac{\mathrm{d}\tau}{\sqrt{(t-\tau)^2 - c^{-2}r^2}}$$

transforms the last integral to

$$\int_{-\infty}^{t-c^{-1}r} \frac{\lambda(\tau)}{2\pi\sqrt{(t-\tau)^2 - c^{-2}r^2}}\mathrm{d}\tau = \int_0^{t-c^{-1}r} \frac{\lambda(\tau)}{2\pi\sqrt{(t-\tau)^2 - c^{-2}r^2}}\mathrm{d}\tau,$$

that is, to the cylindrical (two-dimensional circular) wave from which the layer potential is constructed.

The double layer potential. If we formally take the gradient of G_2

$$\nabla G_2(t,r) = \frac{c^{-1}}{2\pi}\frac{\delta(t-c^{-1}r)}{\sqrt{t^2-c^{-2}r^2}}\,(-\nabla r) - \frac{c^{-2}}{2\pi}\frac{H(t-c^{-1}r)}{(t^2-c^{-2}r^2)^{3/2}}\,(-\tfrac{1}{2}\nabla r^2)$$

we reach an expression for the double layer potential in two dimensions:

$$\begin{aligned}
(\mathcal{D} * \varphi)(\mathbf{x},t) \;:=\; & \frac{c^{-1}}{2\pi}\int_\Gamma \frac{\varphi(\mathbf{u}, t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{\sqrt{(t-\tau)^2 - c^{-2}|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}\Gamma(\mathbf{y}) \\
& - \frac{c^{-2}}{2\pi}\int_\Gamma\int_0^{t-c^{-1}|\mathbf{x}-\mathbf{y}|} \frac{\varphi(\mathbf{y},\tau)}{(t-\tau)^2 - c^{-2}|\mathbf{x}-\mathbf{y}|^2}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{\sqrt{(t-\tau)^2 - c^{-2}|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}\Gamma(\mathbf{y})\mathrm{d}\tau \\
= \; & \int_\Gamma\int_0^t \partial_{\boldsymbol{\nu}(\mathbf{y})} G_2(t-\tau, |\mathbf{x}-\mathbf{y}|)\,\varphi(\mathbf{y},\tau)\mathrm{d}\Gamma(\mathbf{y})\mathrm{d}\tau.
\end{aligned}$$

This one has a circular wave front with no memory plus another one that integrates over the past. Again, it is illustrative to compare with the three dimensional case:

$$\nabla G_3(t,r) = c^{-1}\frac{\delta'(t-c^{-1}r)}{4\pi r}\,(-\nabla r) + \frac{\delta(t-c^{-1}r)}{4\pi r^2}\,(-\nabla r).$$

Chapter 2

From time domain to Laplace domain

A possible theoretical frame for the time domain layer potentials for the wave equation is that of vector valued distributions. From this moment on, instead of thinking of functions of the space and time variables $u(\mathbf{x}, t)$ we will think of functions of the time variable with values on a space of functions of the space variables, which amounts to consider the functions $u(t) = u(\,\cdot\,, t)$ (we will not change the name). In principle, our distributions will be only allowed to take real values, but the test functions (and the Laplace transforms of the distributions) will take complex values.

Some fast references before we take off. The distributional Laplace transform is introduced in most medium/advanced texts on the theory of distributions. The teaching text of Laurent Schwartz [22] is a good introduction. The subject of vector valued distributions is much more involved (and there is the risk of wanting to know and understand it all). François Trèves's book on distributions [23] is the standard (but not easy to read) reference. A compendium of this theory, with a different point of view, can be found in volume five of the English language edition of Robert Dautray and Jacques-Louis Lions's ambitious encyclopedia on theoretical and numerical continuum models [8].

Notation. Given a domain $\mathcal{O}$, we will denote

$$(u, v)_{\mathcal{O}} := \int_{\mathcal{O}} u\, v \qquad (\nabla u, \nabla v)_{\mathcal{O}} := \int_{\mathcal{O}} \nabla u \cdot \nabla v,$$

$$\|u\|_{\mathcal{O}}^2 := (u, \overline{u})_{\mathcal{O}}, \qquad \|\nabla u\|_{\mathcal{O}}^2 := (\nabla u, \nabla \overline{u})_{\mathcal{O}}, \qquad \|u\|_{1,\mathcal{O}}^2 := \|u\|_{\mathcal{O}}^2 + \|\nabla u\|_{\mathcal{O}}^2.$$

We will make unannounced use of basic properties of the Sobolev space $H^1(\mathcal{O})$, the trace operator and the concept of weak normal derivative (we will introduce the latter though). If Γ is a Lipschitz curve/surface, we will use the angled bracket

$$\langle \xi, \eta \rangle_\Gamma := \int_\Gamma \xi\, \eta\, \mathrm{d}\Gamma$$

to denote the $L^2(\Gamma)$-inner product and its extension as the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality product. The $H^{\pm 1/2}(\Gamma)$-norms are denoted $\|\,\cdot\,\|_{\pm 1/2,\Gamma}$. *As a general rule, all brackets will be considered as bilinear (and not sesquilinear) forms, even if spaces include complex valued functions.*

2.1 Causal tempered distributions

We first consider the Schwartz class

$$\mathcal{S}(\mathbb{R}) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}) \,:\, p\,\varphi^{(k)} \in L^\infty(\mathbb{R})\ \forall k \geq 0, \quad \forall p \in \mathcal{P}(\mathbb{R})\},$$

where $\mathcal{P}(\mathbb{R})$ is the space of polynomials with complex coefficients. It is well known that we can define a metric in $\mathcal{S}(\mathbb{R})$ that makes this space complete and such that convergence $\varphi_n \to \varphi$ with respect to that metric is equivalent to convergence in $L^\infty(\mathbb{R})$ of $p\,\varphi_n^{(k)} \to p\,\varphi^{(k)}$ for all $k \geq 0$ and $p \in \mathcal{P}(\mathbb{R})$. It is quite obvious that differentiation and multiplication by polynomials are continuous operators in the Schwartz class.

A tempered distribution with values in the Banach space X is a continuous linear map $f : \mathcal{S}(\mathbb{R}) \to X$. The action of f on a general element $\varphi \in \mathcal{S}(\mathbb{R})$ is denoted using angled brackets $\langle f, \varphi\rangle$. A **causal tempered distribution with values in** X (these are the ones we will be paying attention to) is a tempered X-valued distribution such that

$$\langle f, \varphi\rangle = 0 \qquad \forall \varphi \in \mathcal{S}(\mathbb{R})\,\text{such that}\,\mathrm{supp}\,\varphi \subset (-\infty, 0).$$

There is no commonly used notation for the set of causal tempered X-valued distributions. To avoid continuous repetition of this long expression, we will write

$$f \in \mathrm{CT}(X).$$

Some very simple examples. If $a \in X$, we can define the following two distributions

$$\langle \delta_0 \otimes a, \varphi\rangle := \varphi(0)\,a, \qquad \langle \dot{\delta}_0 \otimes a, \varphi\rangle := -\dot{\varphi}(0)\,a.$$

Some functions are distributions. Many functions can be considered as causal tempered distributions. For instance, if $f : [0, \infty) \to X$ is a continuous function, such that

$$\|f(t)\|_X \leq C\,(1 + t^m) \qquad t \geq 0,$$

we can define

$$\langle f, \varphi\rangle := \int_0^\infty \varphi(t) f(t) \mathrm{d}t. \tag{2.1}$$

(Note that we are giving the same name to f as a function and as a distribution, which is justified by proving that two different functions cannot define the same distribution.) The integral has to be understood in the sense of Bochner, although with the regularity assumed for f, this definition can be done as an improper Riemann integral. The definition (2.1) can be used for $f \in L^p((0, \infty), X)$ for any $1 \leq p \leq \infty$. When functions are used as distributions they are automatically extended by zero for values $t < 0$.

Differentiation. If f is a causal tempered X-valued distribution, we define

$$\langle \dot{f}, \varphi\rangle := -\langle f, \dot{\varphi}\rangle,$$

thus obtaining another element $\dot{f} \in \mathrm{CT}(X)$. (This is very simple to prove.) We will sometimes write $\frac{\mathrm{d}}{\mathrm{d}t} f$ instead of $\dot{f}$. For instance,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\delta_0 \otimes a) = \dot{\delta}_0 \otimes a.$$

We have to be careful with what we understand by differentiation, since this is an operator that considers f as an entity defined for all $t \in \mathbb{R}$ even if it is zero for negative t. If, for instance, we have a function $f \in \mathcal{C}^1([0,\infty), X)$ such that f is bounded and so is its derivative f', then

$$\dot{f} = f' + \delta_0 \otimes f(0),$$

that is, differentiation in the sense of distributions takes into account the jump across the origin.

Composition with steady-state operators. Consider two Banach spaces X and Y and a bounded linear operator $A : X \to Y$. For $f \in \mathrm{CT}(X)$ we define

$$\langle Af, \varphi \rangle := A\langle f, \varphi \rangle$$

and we obtain that $Af \in \mathrm{CT}(Y)$. (Once more, this is very simple to prove.) In particular, if $X \subset Y$ with continuous embedding, every $f \in \mathrm{CT}(X)$ can be understood as $f \in \mathrm{CT}(Y)$.

2.2 Distributional scattered waves

Let Γ be the boundary of a bounded Lipschitz domain $\Omega^- \subset \mathbb{R}^d$. The exterior domain $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ will always be assumed to be connected, just to avoid some minor annoyances. We will however accept the possibility that Ω^- has several connected components, since nothing is changed by this hypothesis.

The domain of the Laplacian. The space

$$H^1_\Delta(\mathbb{R}^d \setminus \Gamma) := \{ u \in H^1(\mathbb{R}^d \setminus \Gamma) \,:\, \Delta u \in L^2(\mathbb{R}^d \setminus \Gamma)\}$$

will be the natural environment for the definition of the layer potentials. The Laplace operator Δ that appears in the definition of this space is the Laplacian in the sense of distributions in $\mathbb{R}^d \setminus \Gamma$. To emphasize this fact, we write $\Delta u \in L^2(\mathbb{R}^d \setminus \Gamma)$, although $L^2(\mathbb{R}^d \setminus \Gamma) \cong L^2(\mathbb{R}^d)$.

The space $H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ is endowed with the norm

$$\|u\|^2_\Delta := \|u\|^2_{\mathbb{R}^d} + \|\nabla u\|^2_{\mathbb{R}^d\setminus\Gamma} + \|\Delta u\|^2_{\mathbb{R}^d\setminus\Gamma},$$

which makes it a Hilbert space. In this set we can define the interior and exterior traces $\gamma^\pm : H^1_\Delta(\mathbb{R}^d \setminus \Gamma) \to H^{1/2}(\Gamma)$ and the interior and exterior normal derivatives $\partial^\pm_\nu : H^1_\Delta(\mathbb{R}^d \setminus \Gamma) \to H^{-1/2}(\Gamma)$. These last ones are defined using Green's formula:

$$\begin{aligned} \langle \partial_\nu u^-, \gamma^- v\rangle_\Gamma &:= (\Delta u, v)_{\Omega^-} + (\nabla u, \nabla v)_{\Omega^-} && \forall v \in H^1(\Omega^-), \\ \langle \partial_\nu u^+, \gamma^+ v\rangle_\Gamma &:= -(\Delta u, v)_{\Omega^+} - (\nabla u, \nabla v)_{\Omega^+} && \forall v \in H^1(\Omega^+). \end{aligned}$$

The jumps of the trace and normal derivative are also well defined (and define bounded operators) in $H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$:

$$[\![\gamma u]\!] = \gamma^- u - \gamma^+ u, \qquad [\![\partial_\nu u]\!] = \partial^-_\nu u - \partial^+_\nu u.$$

Scattered waves around Γ. The set of solutions of the wave equation that we will care about are

$$u \in \mathrm{CT}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \qquad \text{such that} \quad \ddot{u} = \Delta u. \tag{2.2}$$

We need to clarify what is meant by the above equation. We know that $\Delta : H^1_\Delta(\mathbb{R}^d \setminus \Gamma) \to L^2(\mathbb{R}^d \setminus \Gamma)$ is bounded. Therefore, if $u \in \mathrm{CT}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$, then $\Delta u \in \mathrm{CT}(L^2(\mathbb{R}^d \setminus \Gamma))$. As mentioned in Section 2.1, the causal tempered $H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$-valued distribution $\ddot{u}$ can be understood as $\ddot{u} \in \mathrm{CT}(L^2(\mathbb{R}^d \setminus \Gamma))$. Therefore, the equality in (2.2) is that of two causal tempered distributions with values in $L^2(\mathbb{R}^d \setminus \Gamma)$.

A remark concerning initial conditions. The equation (2.2) is implicitly imposing homogeneous initial conditions. If we want to represent a (smooth) function $u : [0, \infty) \to H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ such that $u(0) = u_0$, $u'(0) = v_0$ and $\Delta u(t) = u''(t)$ for all $t \geq 0$ (these time derivatives are strong derivatives), we need to consider a causal distribution u such that

$$\ddot{u} = \Delta u + \delta_0 \otimes v_0 + \dot{\delta}_0 \otimes u_0.$$

This definition includes an extension by zero to $t < 0$ and not a possible extension by solving the wave equation backwards. Once more, in (2.2) we are assuming homogeneous boundary conditions in a sense that is not easy to specify, since there is no time regularity assumed so far.

What we will end up showing. An aim of what follows is to show that solutions of (2.2) are determined (with an explicit formula) by the causal distributions $[\![\gamma u]\!]$ and $[\![\partial_\nu u]\!]$. This is what is known as Kirchhoff's formula in the three dimensional case. We will get to this in Section 3.5.

The exterior Dirichlet problem. An exterior causal solution of the Dirichlet problem can be defined in several different ways. Dirichlet data is $g \in \mathrm{CT}(H^{1/2}(\Gamma))$. We can consider a causal tempered $H^1_\Delta(\Omega^+)$-valued distribution such that

$$\ddot{u} = \Delta u \qquad \text{and} \qquad \gamma u = g,$$

with equalities as elements of $\mathrm{CT}(L^2(\Omega^+))$ and $\mathrm{CT}(H^{1/2}(\Gamma))$. We can also consider u as in (2.2) satisfying the boundary conditions

$$\gamma^+ u = g, \qquad \gamma^- u = 0.$$

(This is a way of taking $u(t) = 0$ in Ω^- for all t, as we will see once we show the corresponding existence and uniqueness results.) A third option consists of looking for u as in (2.2) satisfying

$$\gamma^+ u = \gamma^- u = g.$$

(This will lead to the single-layer representation of the exterior solution.) You might be inclined to think that this is just fooling around with the interior domain, when all we care about is what happens in Ω^+. The truth is that for layer potentials, there is no inside or outside: what layer potentials see is the domain minus the boundary $\mathbb{R}^d \setminus \Gamma$.

2.3 The Laplace transform

In this section we present the Laplace transform for causal tempered distributions. We note that the Laplace transform can be defined for more general distributions than the ones we are going to deal with. The positive half complex plane is denoted

$$\mathbb{C}_+ := \{s \in \mathbb{C} \,:\, \mathrm{Re}\, s > 0\}.$$

Laplace transform of a causal tempered distribution. Given $f \in \mathrm{CT}(X)$ we define the Laplace transform of f as the function

$$\mathbb{C}_+ \ni s \longmapsto \mathrm{F}(s) := \langle f, \exp(-s\,\cdot\,)\rangle. \tag{2.3}$$

It is easy to notice that even if the function $t \mapsto \exp(-s\,t)$ is not an element of the Schwartz class, the definition (2.3) still makes sense because of the causality of f. A fully precise definition of the duality in (2.3) can be done by using a smooth version of the Heaviside function:

$$h \in \mathcal{C}^\infty(\mathbb{R}) \qquad 0 \le h \le 1 \qquad h \equiv 1 \text{ in } [-1/2, \infty), \qquad h \equiv 0 \text{ in } (-\infty, -1].$$

We can then define $\mathrm{F}(s) := \langle f, h\,\exp(-s\,\cdot\,)\rangle$ and prove that the particular choice of h does not modify the definition of the Laplace transform. The Laplace transform of f is also denoted $\mathcal{L}\{f\}$. It is an analytic function of $s \in \mathbb{C}_+$.

Simple properties. The following two Laplace transforms are straightforward to compute

$$\mathcal{L}\{\delta_0 \otimes a\} \equiv a, \qquad \mathcal{L}\{\dot\delta_0 \otimes a\} = s\,a.$$

Also, it is easy to prove that

$$\mathcal{L}\{\dot f\}(s) = s\mathrm{F}(s). \tag{2.4}$$

Again, for smooth bounded functions $f : [0, \infty) \to X$, the differentiation formula (2.4) is taking care of the value at zero. If f' is the distribution associated to the function $f' : [0, \infty) \to X$, (2.4) is just a compact way of writing the much more popular formula (for functions)

$$\mathcal{L}\{f'\}(s) = s\mathrm{F}(s) - f(0).$$

Finally, if $f \in \mathrm{CT}(X)$ and $A \in \mathcal{B}(X, Y)$, then

$$\mathcal{L}\{Af\}(s) = A\,\mathrm{F}(s), \tag{2.5}$$

as can be easily verified.

An important result is the statement of injectivity of the Laplace transform (which is the main reason why it can be used as a transform): *if* $f \in \mathrm{CT}(X)$ *satisfies that* $\mathrm{F}(s) = 0$ *for all* $s \in \mathbb{C}_+$, *then* $f = 0$.

2.4 Potentials in the Laplace domain

A first simple use of the Laplace transform gives uniqueness of solution to some wave propagation problems around Γ.

Proposition 2.4.1. *Let*

$$u \in \mathrm{CT}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \quad \text{such that} \quad \ddot{u} = \Delta u$$

with equality as causal tempered $L^2(\mathbb{R}^d \setminus \Gamma)$-valued distributions. If

$$[\![\gamma u]\!] = 0, \qquad [\![\partial_\nu u]\!] = 0, \tag{2.6}$$

–as $H^{1/2}(\Gamma)$- and $H^{-1/2}(\Gamma)$-valued distributions–, then $u = 0$.

Proof. Note that $\mathrm{U}(s) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ satisfies

$$s^2 \mathrm{U}(s) = \Delta \mathrm{U}(s) \qquad \text{in } \mathbb{R}^d \setminus \Gamma$$

and

$$[\![\gamma \mathrm{U}(s)]\!] = 0 \qquad [\![\partial_\nu \mathrm{U}(s)]\!] = 0$$

for all $s \in \mathbb{C}_+$. (To see this, we just need to apply steady-state bounded operators to the Laplace transform and use (2.5).)

Applying Green's Theorem (the definition of the weak normal derivatives), we easily prove that

$$\int_{\mathbb{R}^d \setminus \Gamma} |\nabla \mathrm{U}(s)|^2 + s^2 |\mathrm{U}(s)|^2 = \langle \partial_\nu^- \mathrm{U}(s), \gamma^- \overline{\mathrm{U}(s)} \rangle_\Gamma - \langle \partial_\nu^+ \mathrm{U}(s), \gamma^+ \overline{\mathrm{U}(s)} \rangle_\Gamma = 0.$$

Therefore

$$0 = \mathrm{Re}\left(\overline{s} \Big(\int_{\mathbb{R}^d \setminus \Gamma} |\nabla \mathrm{U}(s)|^2 + s^2 |\mathrm{U}(s)|^2 \Big) \right) = (\mathrm{Re}\, s) \Big(\|\nabla \mathrm{U}(s)\|^2_{\mathbb{R}^d \setminus \Gamma} + |s|^2 \|\mathrm{U}(s)\|^2_{\mathbb{R}^d \setminus \Gamma} \Big),$$

which implies that $\mathrm{U}(s) = 0$ for all $s \in \mathbb{C}_+$. The injectivity theorem for the Laplace transform implies that $u = 0$. $\square$

Towards a definition of the single layer potential. Formally, the single layer potential $u = \mathcal{S} * \lambda$ is defined as the solution of a transmission problem. If λ is a causal tempered $H^{-1/2}(\Gamma)$-valued distribution, we can think of finding $u \in \mathrm{CT}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$ such that

$$\ddot{u} = \Delta u, \qquad [\![\gamma u]\!] = 0, \qquad [\![\partial_\nu u]\!] = \lambda. \tag{2.7}$$

In the next chapter, when we get to return to the time domain, we will see that problem (2.7) does actually give a correct definition of the retarded single layer potential.

The single layer potential in the Laplace domain. If u is a solution of (2.7), then for all $s \in \mathbb{C}_+$, $\mathrm{U}(s) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ satisfies

$$\Delta \mathrm{U}(s) - s^2 \mathrm{U}(s) = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \qquad [\![\gamma \mathrm{U}(s)]\!] = 0 \qquad [\![\partial_\nu \mathrm{U}(s)]\!] = \Lambda(s). \tag{2.8}$$

However, the solution of (2.8) can be expressed using a single layer potential associated to the differential operator $\Delta - s^2$. For smooth enough densities $\lambda : \Gamma \to \mathbb{C}$ ($\lambda \in L^2(\Gamma)$ is enough), we define

$$\mathrm{S}(s)\lambda := \int_\Gamma E(\,\cdot\,, \mathbf{y}; s)\lambda(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}), \tag{2.9}$$

where

$$E(\mathbf{x}, \mathbf{y}; s) := \begin{cases} \frac{\imath}{4} H_0^{(1)}(\imath s|\mathbf{x} - \mathbf{y}|) & (d = 2), \\ \dfrac{e^{-s\,|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|} & (d = 3), \end{cases} \tag{2.10}$$

is the fundamental solution of $\Delta - s^2$. The integral formula (2.9) can be extended to any $\lambda \in H^{-1/2}(\Gamma)$. For any $s \in \mathbb{C}_+$, $\mathrm{S}(s)$ defines a bounded operator $\mathrm{S}(s) : H^{-1/2}(\Gamma) \to H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ such that $u = \mathrm{S}(s)\lambda$ is the unique solution of

$$\Delta u - s^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \quad [\![\gamma u]\!] = 0, \quad [\![\partial_\nu u]\!] = \lambda.$$

Therefore

$$\mathrm{U}(s) = \mathrm{S}(s)\Lambda(s) \qquad s \in \mathbb{C}_+. \tag{2.11}$$

Once again, we will have to wait until the next section to take an inverse Laplace transform in (2.11) and use it to give a definition of the layer potential in the time domain. For correct definitions of layer potentials and boundary integral operators for elliptic problems, the reader is referred to the very elegant presentation of Martin Costabel in [7]. The entire theory is presented in great detail in the often quoted monograph of Willian McLean on elliptic systems and their boundary integral representations [18].

The single layer operator. If our starting point is the Dirichlet problem:

$$u \in \mathrm{CT}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)), \qquad \ddot{u} = \Delta u, \qquad \gamma^+ u = \gamma^- u = g,$$

where $g \in \mathrm{CT}(H^{1/2}(\Gamma))$, we can use the Laplace transform to note that, if such u exists, then

$$\mathrm{U}(s) = \mathrm{S}(s)\Lambda(s),$$

where $\Lambda : \mathbb{C}_+ \to H^{-1/2}(\Gamma)$ is given at each $s \in \mathbb{C}_+$ as the solution of the boundary integral equation

$$\mathrm{V}(s)\Lambda(s) = \mathrm{G}(s),$$

where

$$H^{-1/2}(\Gamma) \ni \lambda \longmapsto \mathrm{V}(s)\lambda := \gamma^\pm \mathrm{S}(s)\lambda.$$

In Section 2.6 we will give bounds (in terms of s) for the norms $\mathrm{S}(s)$, $\mathrm{V}(s)$ and $\mathrm{V}^{-1}(s)$. These will be used to prove estimates in the time domain, once we have an inversion formula.

Changing the background velocity. In the previous definitions we have taken $c = 1$. It is actually very simple to modify the speed of waves to another value of c. Taking the Laplace transform of $c^{-2}\ddot{u} = \Delta u$, it is clear that we get to the equation $\Delta u - (s/c)^2 u = 0$. Therefore, the layer potential and operator at this velocity are simply $\mathrm{S}(s/c)$ and $\mathrm{V}(s/c)$.

2.5 The energy norm

The key to understanding the single layer potential and operator in the time domain is the correct analysis of the Laplace transformed potential $\mathrm{S}(s)$ and its associated boundary integral operator $\mathrm{V}(s) = \gamma^{\pm}\mathrm{S}(s)$ for every $s \in \mathbb{C}_+$. For short, we will often write

$$\sigma := \operatorname{Re} s.$$

We start with some notation. For $c > 0$, we will write

$$\underline{c} := \min\{1, c\},$$

and we consider the norms

$$|\!|\!| u |\!|\!|_{c,\mathcal{O}}^2 := \|\nabla u\|_{\mathcal{O}}^2 + c^2 \|u\|_{\mathcal{O}}^2.$$

We also consider the bilinear form

$$a_{s,\mathcal{O}}(u, v) := (\nabla u, \nabla v)_{\mathcal{O}} + s^2 (u, v)_{\mathcal{O}}$$

and note the following simple properties:

$$|a_{s,\mathcal{O}}(u, v)| \le |\!|\!| u |\!|\!|_{|s|,\mathcal{O}} |\!|\!| v |\!|\!|_{|s|,\mathcal{O}} \qquad \forall u, v \in H^1(\mathcal{O}), \tag{2.12}$$

$$\operatorname{Re}\big(e^{-\imath \operatorname{Arg} s} a_{s,\mathcal{O}}(u, \overline{u})\big) = \frac{\sigma}{|s|} |\!|\!| u |\!|\!|_{|s|,\mathcal{O}}^2 \qquad \forall u \in H^1(\mathcal{O}). \tag{2.13}$$

Also

$$\underline{\sigma} \|u\|_{1,\mathcal{O}} \le |\!|\!| u |\!|\!|_{|s|,\mathcal{O}} \le \frac{|s|}{\underline{\sigma}} \|u\|_{1,\mathcal{O}} \qquad \forall u \in H^1(\mathcal{O}), \tag{2.14}$$

where we have used that

$$\max\{1, |s|\} \min\{1, \operatorname{Re} s\} \le |s| \qquad \forall s \in \mathbb{C}_+. \tag{2.15}$$

The following careful lifting property, due to Alain Bamberger and Tuong Ha–Duong, appears (in a slightly different language) in [2]. Its proof relies on classical Sobolev space techniques: use a partition of unity, map to a half space and use the Fourier representation of Sobolev spaces in free space.

Proposition 2.5.1. *Let $\mathcal{O}$ be a Lipschitz domain. There exists $C_{\mathcal{O}} > 0$ such that for all $\xi \in H^{1/2}(\partial\mathcal{O})$ and $c > 0$, the solution $u \in H^1(\mathcal{O})$ of the Dirichlet problem*

$$\begin{aligned} -\Delta u + c^2 u &= 0 \qquad \text{in } \mathcal{O}, \\ \gamma u &= \xi \qquad \text{on } \partial\mathcal{O} \end{aligned}$$

can be bounded by

$$|\!|\!| u |\!|\!|_{c,\mathcal{O}} \le C_{\mathcal{O}} \max\{1, c\}^{1/2} \|\xi\|_{1/2,\partial\mathcal{O}}.$$

As a consequence, we obtain a bound for the normal derivative of the solution of $\Delta u - s^2 u = 0$ for $s \in \mathbb{C}_+$: [2, 13].

Proposition 2.5.2. *Let $\mathcal{O}$ be a Lipschitz domain and let $C_{\mathcal{O}} > 0$ be the constant of Proposition 2.5.1. If $s \in \mathbb{C}_+$ and $u \in H^1_\Delta(\mathcal{O})$ satisfies $\Delta u - s^2 u = 0$ in $\mathcal{O}$, then*

$$\|\partial_\nu u\|_{-1/2,\partial\mathcal{O}} \le C_{\mathcal{O}} \left(\frac{|s|}{\underline{\sigma}}\right)^{1/2} |\!|\!| u |\!|\!|_{|s|,\mathcal{O}}.$$

Proof. Let $\xi \in H^{1/2}(\partial\mathcal{O})$ and consider $v \in H^1(\mathcal{O})$ satisfying

$$\Delta v - |s|^2 v = 0, \qquad \gamma v = \xi.$$

Then,

$$\begin{aligned}
|\langle \partial_\nu u, \xi\rangle_{\partial\mathcal{O}}| &= \big|(\nabla u, \nabla v)_{\mathcal{O}} + s^2 (u,v)_{\mathcal{O}}\big| && \text{(since } \Delta u - s^2 u = 0\text{)}\\
&= |a_{s,\mathcal{O}}(u,v)| \\
&\le |\!|\!| u |\!|\!|_{|s|,\mathcal{O}} |\!|\!| v |\!|\!|_{|s|,\mathcal{O}} && \text{(by (2.12))}\\
&\le C_{\mathcal{O}} \max\{1,|s|\}^{1/2} |\!|\!| u |\!|\!|_{|s|,\mathcal{O}} \|\xi\|_{1/2,\partial\mathcal{O}} && \text{(by Proposition 2.5.1)}\\
&\le C_{\mathcal{O}} \left(\frac{|s|}{\underline{\sigma}}\right)^{1/2} |\!|\!| u |\!|\!|_{|s|,\mathcal{O}} \|\xi\|_{1/2,\partial\mathcal{O}}. && \text{(by (2.15))}
\end{aligned}$$

Therefore

$$\|\partial_\nu u\|_{-1/2,\partial\mathcal{O}} = \sup_{0\ne\xi\in H^{1/2}(\partial\mathcal{O})} \frac{|\langle \partial_\nu u, \xi\rangle_{\partial\mathcal{O}}|}{\|\xi\|_{1/2,\partial\mathcal{O}}} \le C_{\mathcal{O}} \left(\frac{|s|}{\underline{\sigma}}\right)^{1/2} |\!|\!| u |\!|\!|_{|s|,\mathcal{O}},$$

and the proof is finished. □

2.6 Bounds in the resolvent set

We start with an ellipticity estimate [2, 13] for $\mathrm{V}(s)$. A time domain version of it will be given in Section 3.7.

Proposition 2.6.1. *There exists $C_\Gamma > 0$ such that for all $s \in \mathbb{C}_+$*

$$\mathrm{Re}\left(e^{\imath \mathrm{Arg}\, s} \langle \overline{\lambda}, \mathrm{V}(s)\lambda\rangle_\Gamma\right) \ge C_\Gamma \frac{\sigma\underline{\sigma}}{|s|^2} \|\lambda\|^2_{-1/2,\Gamma} \qquad \forall \lambda \in H^{-1/2}(\Gamma).$$

Therefore $\mathrm{V}(s) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible and

$$\|\mathrm{V}(s)^{-1}\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \le C_\Gamma^{-1} \frac{|s|^2}{\sigma\underline{\sigma}}.$$

Proof. The proof of this result relies on the results of Section 2.5 together with the understanding of $\mathrm{V}(s)$ in terms the solution of a transmission problem, a technique that goes back (at least) to the pioneering work of Jean–Claude Nédélec and Jacques Planchard [19] for the single layer operator of the Laplace equation. Let $\lambda \in H^{-1/2}(\Gamma)$ and $u = \mathrm{S}(s)\lambda$. Then $u \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ is the unique solution of the transmission problem

$$\Delta u - s^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \qquad [\![\gamma u]\!] = 0, \qquad [\![\partial_\nu u]\!] = \lambda \tag{2.16}$$

and

$$\mathrm{V}(s)\lambda = \gamma^\pm u. \tag{2.17}$$

Therefore

$$\begin{aligned}
\langle \overline{\lambda}, \mathrm{V}(s)\lambda\rangle_\Gamma &= \langle \partial_\nu^- \overline{u}, \gamma^- u\rangle_\Gamma - \langle \partial_\nu^+ \overline{u}, \gamma^+ u\rangle_\Gamma && \text{(by (2.16) and (2.17))}\\
&= (\nabla \overline{u}, \nabla u)_{\mathbb{R}^d\setminus\Gamma} + \overline{s}^2 (\overline{u}, u)_{\mathbb{R}^d\setminus\Gamma} && (\text{since } \Delta\overline{u} - \overline{s}^2\overline{u} = 0)\\
&= a_{\overline{s},\mathbb{R}^d\setminus\Gamma}(\overline{u}, u).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Re}\big(e^{\imath \mathrm{Arg} s}\langle \overline{\lambda}, \mathrm{V}(s)\lambda\rangle_\Gamma\big) &= \mathrm{Re}\big(e^{-\imath \mathrm{Arg}\overline{s}} a_{\overline{s},\mathbb{R}^d\setminus\Gamma}(u, \overline{u})\big)\\
&= \frac{\sigma}{|s|} |\!|\!| u |\!|\!|^2_{|s|,\mathbb{R}^d\setminus\Gamma} && \text{(by (2.13))}\\
&\ge \frac{\sigma}{|s|}\frac{\underline{\sigma}}{|s|}(C^2_{\Omega^-}\|\partial_\nu^- u\|^2_{-1/2,\Gamma} + C^2_{\Omega^+}\|\partial_\nu^+ u\|^2_{-1/2,\Gamma}) && \text{(by Prop 2.5.2)}\\
&\ge C_\Gamma \frac{\sigma\underline{\sigma}}{|s|^2}\|\lambda\|^2_{-1/2,\Gamma} && (\lambda = [\![\partial_\nu u]\!]).
\end{aligned}$$

This coercivity estimate proves invertibility of $\mathrm{V}(s)$ by a Lax-Milgram argument and a bound for the inverse of $\mathrm{V}(s)$. $\square$

Proposition 2.6.2. *There exists $C_\Gamma > 0$ such that for all $s \in \mathbb{C}_+$*

$$\|\mathrm{S}(s)\lambda\|_{1,\mathbb{R}^d\setminus\Gamma} \le C_\Gamma \frac{|s|}{\sigma\underline{\sigma}^2}\|\lambda\|_{-1/2,\Gamma} \qquad \forall \lambda \in H^{-1/2}(\Gamma) \tag{2.18}$$

and

$$\|\mathrm{V}(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C_\Gamma^2 \frac{|s|}{\sigma\underline{\sigma}^2}. \tag{2.19}$$

Finally, there exists $C'_\Gamma > 0$ such that for all $s \in \mathbb{C}_+$,

$$\|\partial_\nu^\pm \mathrm{S}(s)\lambda\|_{-1/2,\Gamma} \le C'_\Gamma \frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}}\|\lambda\|_{-1/2,\Gamma} \qquad \forall \lambda \in H^{-1/2}(\Gamma). \tag{2.20}$$

Proof. Let $\lambda \in H^{-1/2}(\Gamma)$ and $u = \mathrm{S}(s)\lambda$. Then

$$\begin{aligned}
\frac{\sigma}{|s|}\underline{\sigma}^2\|u\|^2_{1,\mathbb{R}^d\setminus\Gamma} &\le \frac{\sigma}{|s|}|\!|\!| u |\!|\!|^2_{|s|,\mathbb{R}^d\setminus\Gamma} && \text{(by (2.14))}\\
&= \mathrm{Re}\left(e^{\imath \mathrm{Arg}\, s}\langle\overline{\lambda}, \mathrm{V}(s)\lambda\rangle_\Gamma\right) && \text{(proof of Prop 2.6.1)}\\
&\le \|\lambda\|_{-1/2,\Gamma}\|\gamma u\|_{1/2,\Gamma} && (\gamma u = \mathrm{V}(s)\lambda)\\
&\le C_\Gamma \|\lambda\|_{-1/2,\Gamma}\|u\|_{1,\mathbb{R}^d\setminus\Gamma}. && (C_\Gamma = \|\gamma\|)
\end{aligned}$$

This proves (2.18) and therefore (2.19). Note finally that

$$\begin{aligned}
\|\partial_\nu^\pm u\|^2_{-1/2,\Gamma} &\le D_\Gamma \frac{|s|}{\underline{\sigma}} \|\!| u |\!\|^2_{|s|,\mathbb{R}^d\setminus\Gamma} && \text{(by Proposition 2.5.2)}\\
&\le C_\Gamma D_\Gamma \frac{|s|}{\underline{\sigma}} \frac{|s|}{\sigma} \|\lambda\|_{-1/2,\Gamma} \|\mathrm{V}(s)\lambda\|_{1/2,\Gamma} && \text{(see above)}\\
&\le C_\Gamma^2 D_\Gamma \frac{|s|^3}{\sigma^2\underline{\sigma}^3} \|\lambda\|^2_{-1/2,\Gamma} && \text{(by (2.19))},
\end{aligned}$$

which completes the proof. □

Proposition 2.6.3. *For all $s \in \mathbb{C}_+$,*

$$\|\mathrm{S}(s)\lambda\|_\Delta \le 2C_\Gamma \frac{|s|^2}{\sigma\underline{\sigma}^3} \|\lambda\|_{-1/2,\Gamma} \qquad \forall \lambda \in H^{-1/2}(\Gamma),$$

where C_Γ is the constant of Proposition 2.6.2.

Proof. Let $u = \mathrm{S}(s)\lambda$. Then, by the arguments of the proof of Proposition 2.6.1 and by Proposition 2.6.2,

$$\|\!| u |\!\|^2_{|s|,\mathbb{R}^d\setminus\Gamma} \le \frac{|s|}{\sigma} |\langle \overline{\lambda}, \mathrm{V}(s)\lambda\rangle_\Gamma| \le C_\Gamma^2 \frac{|s|^2}{\sigma^2\underline{\sigma}^2} \|\lambda\|^2_{-1/2,\Gamma}.$$

Therefore, since $\Delta u = s^2 u$, we can bound

$$\|\Delta u\|_{\mathbb{R}^d\setminus\Gamma} = \|s^2 u\|_{\mathbb{R}^d\setminus\Gamma} \le |s| \|\!| u |\!\|_{|s|,\mathbb{R}^d\setminus\Gamma} \le C_\Gamma \frac{|s|^2}{\sigma\underline{\sigma}} \|\lambda\|_{-1/2,\Gamma}. \tag{2.21}$$

The result is then a consequence of (2.18) and (2.21) with the overestimate $\frac{1}{\underline{\sigma}} + |s| \le 2\frac{|s|}{\underline{\sigma}^2}$. □

2.7 Appendix: Proof of the lifting lemma

In this section we give a proof of Proposition 2.5.1. This is done in three steps: lifting in the half space, lifting in the half space with compact support, and mapping to a general domain. We will denote

$$\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0,\infty), \qquad \partial_\circ\mathbb{R}^d = \mathbb{R}^{d-1} \times \{0\},$$

and $\gamma_\circ : H^1(\mathbb{R}^d_+) \to H^{1/2}(\mathbb{R}^{d-1})$ for the trace operator. We will write $\mathbb{R}^d \ni \mathbf{x} = (\widetilde{\mathbf{x}}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$,

Lemma 2.7.1. *Let $\xi \in H^{1/2}(\mathbb{R}^{d-1}) \equiv H^{1/2}(\partial_\circ\mathbb{R}^d)$ and let $w \in H^1(\mathbb{R}^d_+)$ be the solution of*

$$-\Delta w + c^2 w = 0 \qquad \text{in } \mathbb{R}^d_+, \tag{2.22a}$$

$$\gamma_\circ w = \xi \qquad \text{on } \partial_\circ\mathbb{R}^d. \tag{2.22b}$$

Then

$$\|\nabla w\|^2_{\mathbb{R}^d_+} + c^2\|w\|^2_{\mathbb{R}^d_+} \le (2\pi)^{d-1} \max\{1, c\} \|\xi\|^2_{1/2,\mathbb{R}^{d-1}}.$$

Proof. Using a density argument (for smooth functions) it is easy to show that the $(d-1)$-variable Fourier transform

$$\breve{w}(\boldsymbol{\omega}, x_d) := \int_{\mathbb{R}^{d-1}} e^{-\imath \boldsymbol{\omega}\cdot\widetilde{\mathbf{x}}} w(\widetilde{\mathbf{x}}, x_d) \mathrm{d}\widetilde{\mathbf{x}}$$

transforms $L^2(\mathbb{R}^d_+)$ into itself. Again with a density argument, we can prove that the solution of (2.22) is in the anisotropic Sobolev space $\breve{H} := \{v \in L^2(\mathbb{R}^d_+) \,:\, \partial_{x_d} v \in L^2(\mathbb{R}^d_+)\}$ and satisfies

$$\begin{aligned} -\partial_{x_d x_d}\breve{w} + (|\boldsymbol{\omega}|^2 + c^2)\breve{w} = 0 \qquad & \text{in } \mathbb{R}^d_+, \\ \breve{\gamma}_0 \breve{w} = \widehat{\xi} \qquad & \text{on } \partial_\circ \mathbb{R}^d, \end{aligned}$$

where $\breve{\gamma}_\circ : \breve{H} \to L^2(\mathbb{R}^{d-1})$ is the associated one-dimensional trace operator and $\widehat{\xi}$ is the Fourier transform of ξ. Therefore

$$\breve{w}(\boldsymbol{\omega}, x_d) = \widehat{\xi}(\boldsymbol{\omega}) e^{-x_d(|\boldsymbol{\omega}|^2 + c^2)^{1/2}}.$$

In other words, we can prove that

$$w(\widetilde{\mathbf{x}}, x_d) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{\imath \boldsymbol{\omega}\cdot\widetilde{\mathbf{x}}} \widehat{\xi}(\boldsymbol{\omega}) e^{-x_d(|\boldsymbol{\omega}|^2+c^2)^{1/2}} \mathrm{d}\boldsymbol{\omega}$$

is the solution operator corresponding to problem (2.22). Using the Plancherel identity corresponding to the transform $w \mapsto \breve{w}$, we can easily justify the following:

$$\begin{aligned} &\|\nabla w\|^2_{\mathbb{R}^d_+} + c^2\|w\|^2_{\mathbb{R}^d_+} \\ &\quad = (2\pi)^{d-1} \int_{\mathbb{R}^d_+} \Big(|\partial_{x_d}\breve{w}(\boldsymbol{\omega}, x_d)|^2 + (c^2 + |\boldsymbol{\omega}|^2)|\breve{w}(\boldsymbol{\omega}, x_d)|^2 \Big) \mathrm{d}\boldsymbol{\omega}\mathrm{d}x_d \\ &\quad = (2\pi)^{d-1} \int_{\mathbb{R}^d_+} \Big(2(c^2 + |\boldsymbol{\omega}|^2)|\widehat{\xi}(\boldsymbol{\omega})|^2 e^{-2x_d(c^2+|\boldsymbol{\omega}|^2)^{1/2}} \Big) \mathrm{d}\boldsymbol{\omega}\mathrm{d}x_d \\ &\quad = (2\pi)^{d-1} \int_{\mathbb{R}^{d-1}} (c^2 + |\boldsymbol{\omega}|^2)^{1/2} |\widehat{\xi}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega} \\ &\quad \le (2\pi)^{d-1} \max\{1, c\} \int_{\mathbb{R}^{d-1}} (1 + |\boldsymbol{\omega}|^2)^{1/2} |\widehat{\xi}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega} \\ &\quad = (2\pi)^{d-1} \max\{1, c\} \|\xi\|^2_{1/2,\mathbb{R}^{d-1}}. \end{aligned}$$

Note that the last equality is due to the Fourier transform representation of the space $H^{1/2}(\mathbb{R}^{d-1})$. □

Lemma 2.7.2. *Let $r < 1$ and*

$$H^{1/2}_r(\mathbb{R}^{d-1}) := \{\xi \in H^{1/2}(\mathbb{R}^{d-1}) \,:\, \operatorname{supp}\xi \subset \overline{B(\mathbf{0}; r)}\}.$$

Consider the following sets

$$\mho := B(\mathbf{0}; 1) \times (-1, 1) \subset \mathbb{R}^d, \quad \mho_+ := \mho \cap \mathbb{R}^d_+, \quad \partial_\circ \mho := \mho \cap \partial_\circ \mathbb{R}^d = B(\mathbf{0}; 1) \times \{0\},$$

and the Sobolev space $H^1_\star(\mho_+) := \{u \in H^1(\mho_+) \,:\, \gamma u = 0 \text{ in } \partial\mho_+ \setminus \partial_\circ \mho\}$. *For every* $c > 0$, *there exists a bounded operator* $\gamma_\circ^+ : H_r^{1/2}(\mathbb{R}^{d-1}) \to H^1_\star(\mho_+)$ *satisfying*

$$\gamma_\circ \gamma_\circ^+ \xi = \xi \qquad \forall \xi \in H_r^{1/2}(\mathbb{R}^{d-1})$$

and

$$\|\nabla v\|^2_{\mho_+} + c^2 \|v\|^2_{\mho_+} \le C \,\max\{1, c\} \|\xi\|^2_{1/2,\mathbb{R}^{d-1}}, \quad v = \gamma_\circ^+ \xi$$

where $C > 0$ *is independent of* c.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ satisfy: $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B(\mathbf{0}; r) \times (-1/2, 1/2)$ and $\operatorname{supp} \varphi \subset \mho$ if. We then define $\gamma_\circ^+ \xi := \varphi w$, where w solves (2.22) with $\max\{1, c\}$ instead of c. The rest of the proof is left as an exercise. $\square$

Lemma 2.7.3. *Let* $\mathcal{O}$ *be a Lispchitz domain. For all* $c > 0$ *there exists an operator* $\gamma^+ : H^{1/2}(\partial\mathcal{O}) \to H^1(\mathcal{O})$ *such that*

$$\gamma \gamma^+ \xi = \xi \quad \forall \xi \in H^{1/2}(\partial\mathcal{O})$$

and

$$\|\nabla v\|^2_{\mathcal{O}} + c^2 \|v\|^2_{\mathcal{O}} \le C \max\{1, c\} \|\xi\|^2_{1/2,\partial\Omega} \qquad v = \gamma^+ \xi. \tag{2.23}$$

Proof. Let $\mho, \mho_+$ and $\partial_\circ \mho$ be as in Lemma 2.7.2. We can then find open sets Ω_j and Lipschitz maps with Lipschitz inverses $\chi_j : \mho \to \Omega_j$ such that $\{\Omega_1, \ldots, \Omega_M\}$ is a cover of $\partial\Omega$ with χ_j mapping $\mho_+$ into $\Omega_j \cap \mathcal{O}$ and $\partial_\circ \mho$ into $\Omega_j \cap \partial\mathcal{O}$. To this covering of $\partial\mathcal{O}$ we associate smooth compactly supported functions $\{\varphi_1, \ldots, \varphi_m\}$ such that $\operatorname{supp} \varphi_j \subset \Omega_j$ and $\sum_j \varphi_j \equiv 1$ in a neighborhood of $\partial\mathcal{O}$. The functions $\widetilde{\varphi}_j := \varphi_j \circ \chi_j|_{\partial_\circ \mho}$ are compactly supported in $B(\mathbf{0}; 1) \equiv \partial_\circ \mho_+$ and we can therefore find $r > 0$ such that

$$\operatorname{supp} \widetilde{\varphi}_j \subset \overline{B(\mathbf{0}; r)} \quad j = 1, \ldots, M.$$

Therefore, the functions

$$\widetilde{\xi}_j := \widetilde{\varphi}_j (\xi \circ \chi_j|_{\partial_\circ \mho}) = (\varphi_j \, \xi) \circ \chi_j|_{\partial_\circ \mho}$$

can be extended by zero to $\mathbb{R}^{d-1} \setminus B(\mathbf{0}; 1)$ and are thus elements of the space $H_r^{1/2}(\mathbb{R}^{d-1})$ in Lemma 2.7.2. Finally we define

$$\gamma^+ \xi := \sum_{j=1}^M (\gamma_\circ^+ \widetilde{\xi}_j) \circ \Xi_j, \tag{2.24}$$

where $\gamma_\circ^+$ is the lifting of Lemma 2.7.2, $\Xi_j : \Omega_j \cap \mathcal{O} \to \mho_+$ is the inverse of $\chi_j|_{\mho_+}$ and the functions $(\gamma_\circ^+ \widetilde{\xi}_j) \circ \Xi_j : \Omega_j \cap \mathcal{O} \to \mathbb{R}$ are extended by zero to the rest of $\mathcal{O}$. The rest of the proof is left as an exercise. $\square$

Proof of Proposition 2.5.1. The solution of the Dirichlet problem

$$\begin{aligned} -\Delta u + c^2 u = 0 \qquad & \text{in } \mathcal{O}, \\ \gamma u = \xi \qquad & \text{on } \partial\mathcal{O} \end{aligned}$$

minimizes the quadratic functional $\|\nabla u\|^2_{\mathcal{O}} + c^2 \|u\|^2_{\mathcal{O}}$ among all elements of $H^1(\mathcal{O})$ such that $\gamma u = \xi$. Therefore, the result is a straightforward consequence of Lemma 2.7.3. $\square$

2.8 Exercises

1. (Section 2.1) **Tensor product distributions.** Let $T \in \mathcal{S}'(\mathbb{R})$ be any tempered scalar distribution and let $a \in X$, where X is a Banach space. Show that $u := T \otimes a$ defined by
$$\langle T \otimes a, \varphi \rangle := \langle T, \varphi \rangle_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})}\, a$$
is a tempered x-valued distribution.

2. (Section 2.1) **A paradoxical example of distributions**.

   (a) Let χ be the characteristic function of the interval $(0,1)$. Show that the traveling wave function u
   $$u(t) := \chi(\cdot - t) : \mathbb{R} \to \mathbb{R}$$
   is a tempered $H^2(\mathbb{R})$ valued distribution, even if as a function it never takes values on $H^2(\mathbb{R})$. **Hint.** Note that $\langle u, \varphi \rangle = \chi * \varphi$, where $*$ denotes convolution in $\mathbb{R}$.

   (b) Extend the previous example to any $\chi \in L^1(\mathbb{R})$ or $\chi \in L^2(\mathbb{R})$.

   (c) Show that
   $$\ddot{u} = \partial_x^2 u$$
   as tempered distributions with values in $L^2(\mathbb{R})$. (Here $\partial_x^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the distributional derivative that acts as a steady-state operator.) **Hint.** Note that by definition $\langle \partial_x^2 u, \varphi \rangle = \partial_x^2 \langle u, \varphi \rangle$.

3. (Section 2.1) **Standing functions.** Let $\chi \in H^2(\mathbb{R})$ and define $u(t) := \chi$ for all t. Show that u is a tempered $H^2(\mathbb{R})$-valued dsitribution. Write u as a tensor product. Show that if $\chi \notin H^2(\mathbb{R})$, then u is not an $H^2(\mathbb{R})$-valued distribution.

4. (Section 2.1) **Time-harmonic waves.** Let $\omega > 0$. Consider the function $\phi_\omega(t) := \exp(-\imath\omega t)$ and $v \in H^1_\Delta(\Omega)$ satisfy $\Delta v + \omega^2 v = 0$. Show that the $H^1_\Delta(\Omega)$-valued distribution $u := \phi_\omega \otimes v$ satisfies the wave equation
$$\ddot{u} = \Delta u,$$
with equality as tempered (non-casual) distributions with values in $L^2(\Omega)$.

5. (Section 2.1) **The passage of a plane wave.** Consider a function $\chi \in L^\infty(\mathbb{R})$, a bounded domain $\Omega \in \mathbb{R}^d$ and a fixed direction $\mathbf{d} \in \mathbb{R}^d$ (with $|\mathbf{d}| = 1$). For $t \in \mathbb{R}$, we define $u(t) : \Omega \to \mathbb{R}$ as
$$u(t)(\mathbf{x}) := \chi(\mathbf{x} \cdot \mathbf{d} - t) \qquad \mathbf{x} \in \Omega.$$

   (a) Show that u is an $H^2(\Omega)$-valued tempered distribution. (Note that in principle u is not an $H^2(\mathbb{R}^d)$-valued distribution, unless $d = 1$ and χ satisfies some integrability conditions.)

   (b) Show that $\ddot{u} = \Delta u$, with equality as tempered distributions with values in $L^2(\Omega)$.

(c) Give conditions on χ and Ω that ensure that u is causal.

6. (Section 2.1) Show that if $f \in \mathrm{CT}(X)$ is causal and $\operatorname{supp} \dot{f} \subset [T,\infty)$, then $\operatorname{supp} f \subset [T,\infty)$. (**Hint.** The proof is the same as for scalar valued distributions. Choose a fixed $\varphi_0 \in \mathcal{D}(-\infty,T)$ such that $\int_{-\infty}^T \varphi_0 = 1$, and use the process

$$\mathcal{D}(-\infty,T) \ni \psi \longmapsto \varphi(t) := \int_{-\infty}^t \Big(\psi(\tau) - \Big(\int_{-\infty}^T \psi\Big)\varphi_0(\tau)\Big)\mathrm{d}\tau \in \mathcal{D}(-\infty,T)$$

to show that $\langle f, \psi\rangle = \int_{-\infty}^T \psi \langle f, \varphi_0\rangle = 0$).

7. (Section 2.7) Complete the proof of Lemma 2.7.2.

$$\|\nabla(\varphi w)\|^2_{\mho_+} + c^2\|\varphi w\|^2_{\mho_+} \le 2\|\nabla w\|^2_{\mho_+} + (c^2 + 2\|\nabla\varphi\|^2_{L^\infty(\mho_+)})\|w\|^2_{\mho_+}.$$

This justifies why to use $\max\{1,c\}$ instead of c in (2.22).

8. (Section 2.7) Complete the proof of Lemma 2.7.3. Use the fact that

$$\|\nabla(w\circ\chi_j)\|^2_{\Omega_j\cap\mathcal{O}} + c^2\|w\circ\chi_j\|^2_{\Omega_j\cap\mathcal{O}} \le C\Big(\|\nabla w\|^2_{\mho_+} + c^2\|w\|^2_{\mho_+}\Big) \qquad \forall w \in H^1(\mho_+).$$

Justify that the function defined in (2.24) is well defined as a function in $H^1(\Omega)$, that its trace is ξ and that the bound (2.23) is satisfied.

Chapter 3

From Laplace domain to time domain

The task of identifying what functions $\mathrm{F}(s)$ are Laplace transforms of causal functions (distributions) is part of what is often called the Paley-Wiener Theorem, which is a collection of results related to holomorphic extensions of the Fourier transform that can be understood as two-sided Laplace transforms. Our presentation will first restrict the kind of symbols (Laplace transforms) that we want to invert. Part of the material that follows is adapted (and modified) from the PhD dissertation of Antonio Laliena [11].

3.1 Inversion of the Laplace transform

A class of symbols. Let X be a Banach space and $\mu \in \mathbb{R}$. We write $\mathrm{F} \in \mathcal{A}(\mu, X)$ when F is an analytic function

$$\mathrm{F} : \mathbb{C}_+ \to X$$

such that

$$\|\mathrm{F}(s)\| \le C_{\mathrm{F}}(\operatorname{Re} s)|s|^\mu \qquad \forall s \in \mathbb{C}_+$$

where $C_{\mathrm{F}} : (0, \infty) \to (0, \infty)$ is a non-increasing function such that

$$C_{\mathrm{F}}(\sigma) \le \frac{C}{\sigma^m} \qquad \forall \sigma \in (0, 1]. \tag{3.1}$$

Since $1 \le |s|/(\operatorname{Re} s)$, it is clear that $\mathcal{A}(\mu, X) \subset \mathcal{A}(\mu + \varepsilon, X)$ for all $\varepsilon > 0$.

This class of symbols is a further refinement of a class introduced by Antonio Laliena and myself in [13] motivated by the potentials and operators on the resolvent set (see Section 2.6) and with a view on the analysis of Convolution Quadrature methods (see Chapter 4). The class is more restrictive than the one dealt with by Christian Lubich in his introduction to Convolution Quadrature methods for hyperbolic problems [16]. In

Section 2.6, we have seen that

$$\begin{array}{rcl} \mathrm{S} & \in & \mathcal{A}(1, \mathcal{B}(H^{-1/2}(\Gamma), H^1(\mathbb{R}^d \setminus \Gamma)), \\ \mathrm{S} & \in & \mathcal{A}(2, \mathcal{B}(H^{-1/2}(\Gamma), H^1_\Delta(\mathbb{R}^d \setminus \Gamma)), \\ \mathrm{V} & \in & \mathcal{A}(1, \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)), \\ \mathrm{V}^{-1} & \in & \mathcal{A}(2, \mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)), \\ \mathrm{K}^t & \in & \mathcal{A}(\tfrac{3}{2}, \mathcal{B}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma)), \end{array}$$

where

$$\mathrm{K}^t(s) := \{\!\!\{\partial_\nu \mathrm{S}(s)\}\!\!\} = \tfrac{1}{2}(\partial_\nu^- \mathrm{S}(s) + \partial_\nu^+ \mathrm{S}(s)).$$

The strong inversion formula. Let $\mathrm{F} \in \mathcal{A}(\mu, X)$ with $\mu < -1$. For any $\sigma > 0$ we define

$$f(t) := \frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{s\,t}\mathrm{F}(s)\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\sigma+\imath\omega)\,t}\mathrm{F}(\sigma+\imath\omega)\,\mathrm{d}\omega. \tag{3.2}$$

A simple computation shows that $f(t)$ is well defined for all $t \in \mathbb{R}$ and that

$$\|f(t)\| \le \frac{1}{2\pi} C_\mathrm{F}(\sigma)\sigma^{1+\mu} e^{\sigma t} B\Big(\tfrac{1}{2}, -\tfrac{\mu+1}{2}\Big), \tag{3.3}$$

where B is the Euler beta function. A classical (but non-trivial) contour integration argument can then be used to prove that $f(t)$ is actually independent of $\sigma > 0$. Taking the limit as $\sigma \to \infty$ in (3.3), we can prove that $f(t) = 0$ for $t \le 0$. For $t \ge 1$, we can use (3.3) with $\sigma = t^{-1}$ and (3.1) to obtain that

$$\|f(t)\| \le \frac{1}{2\pi} C_\mathrm{F}(t^{-1}) t^{|1+\mu|} B\Big(\tfrac{1}{2}, -\tfrac{\mu+1}{2}\Big) \le C\, t^{m+|1+\mu|} B\Big(\tfrac{1}{2}, -\tfrac{\mu+1}{2}\Big),$$

which ensures polynomial growth of $f(t)$ for large t. The Dominated Convergence Theorem can also be used to prove that f is a continuous function of t. Therefore f is a causal tempered distribution: $f \in \mathrm{CT}(X)$. Finally, it can be proved that $\mathcal{L}\{f\} = \mathrm{F}$. In summary, we have sketched the proof of the following result:

Proposition 3.1.1. *If* $\mathrm{F} \in \mathcal{A}(\mu, X)$ *with* $\mu < -1$, *then* F *is the Laplace transform of a continuous causal function* $f : \mathbb{R} \to X$ *with polynomial growth.*

The general case is a consequence of the formula $\mathcal{L}\{\phi^{(k)}\} = s^k \mathcal{L}\{\phi\}$, since $\mathrm{F} \in \mathcal{A}(\mu, X)$ implies that $s^{-k}\mathrm{F} \in \mathcal{A}(\mu - k, X)$. Proposition 3.1.1 can then be invoked for the symbol $s^{-k}\mathrm{F}(s)$ for sufficiently large k to obtain the following result:

Proposition 3.1.2. *Let* $\mathrm{F} \in \mathcal{A}(\mu, X)$ *with* $\mu \in \mathbb{R}$ *and let* $k := \min\{0, \lfloor \mu + 1 \rfloor\}$. *Then there exists a continuous causal function* $\phi : \mathbb{R} \to X$ *with polynomial growth such that* $\mathrm{F} = \mathcal{L}\{\phi^{(k)}\}$.

The set of all causal tempered distributions whose Laplace transforms are elements of $\mathcal{A}(\mu, X)$ for some μ can be studied similarly, starting with a simple case and proceeding by differentiation. The key idea is the fact that for a continuous causal function such that

$$\|f(t)\| \le C(1 + t^m) \qquad \forall t \ge 0,$$

we can use the integral form of the Laplace transform and bound

$$\|\mathrm{F}(s)\| \le C \int_0^\infty e^{-t\mathrm{Re}\, s}(1+t^m)\mathrm{d}t = C\left(\frac{1}{\mathrm{Re}\, s} + \frac{m!}{(\mathrm{Re}\, s)^{m+1}}\right).$$

Proposition 3.1.3. *If $f:\mathbb{R}\to X$ is a continuous causal function with polynomial growth, then $\mathrm{F}\in\mathcal{A}(0,X)$. Therefore the set of symbols*

$$\bigcup_{\mu\in\mathbb{R}} \mathcal{A}(\mu, X)$$

is the set of the Laplace transforms of continuous causal functions $\mathbb{R}\to X$ with polynomial growth and their distributional derivatives.

For short, the set of all possible distributional derivatives of continuous causal X-valued functions with polynomial growth will be given the notation $\mathrm{TD}(X)$ (TD as in time domain). This is non-standard temporary notation, waiting to be improved.

3.2 From symbols to convolution operators

Let X and Y be two Banach spaces. Let $a\in\mathrm{TD}(\mathcal{B}(X,Y))\subset\mathrm{CT}(\mathcal{B}(X,Y))$ be such that $\mathrm{A}=\mathcal{L}\{a\}\in\mathcal{A}(\mu,\mathcal{B}(X,Y))$. Some very general results on the theory of vector valued distributions can be invoked to prove that given any causal (not necessarily tempered) X-valued distribution g, the convolution $a*g$ is well defined as a causal Y-valued distribution. Even if g is tempered, it does not follow that $a*g$ is tempered. However, the class $\mathrm{TD}(X)$ is mapped to the class $\mathrm{TD}(Y)$ by convolution with a and we will use this fact to give an alternative (equivalent) definition of convolution.

If $a\in\mathrm{TD}(\mathcal{B}(X,Y))$ and $g\in\mathrm{TD}(X)$, we define

$$a*g := \mathcal{L}^{-1}\{\mathrm{A}\,\mathrm{G}\}. \tag{3.4}$$

It is easy to see that this definition makes sense by noticing that if $\mathrm{G}\in\mathcal{A}(\nu,X)$, then

$$\|\mathrm{A}(s)\mathrm{G}(s)\|_Y \le \|\mathrm{A}(s)\|_{\mathcal{L}(X,Y)}\|\mathrm{G}(s)\|_X \le C_{\mathrm{A}}(\mathrm{Re}\, s)C_{\mathrm{G}}(\mathrm{Re}\, s)|s|^{\mu+\nu}.$$

An important detail concerns the preservation of some kind of delayed causality, namely, we want to prove that if $g=0$ in $(-\infty,T)$, then so is $a*g$. For any $T\in\mathbb{R}$, we can define the translated distribution $g_T := g(\,\cdot\,-T)$ by

$$\langle g_T,\varphi\rangle := \langle g,\varphi(\,\cdot\,+T)\rangle.$$

If $g\in\mathrm{CT}(X)$ and $T>0$, then $g_T\in\mathrm{CT}(X)$ (causality is not lost by a displacement to the right), but in general g_{-T} will not be causal. The assertion that g_{-T} is causal is an equivalent way of saying that $g=0$ in $(-\infty,T)$.

Proposition 3.2.1. *Let $T>0$ and $g\in\mathrm{TD}(X)$ be such that $\mathrm{supp}\, g\subset[T,\infty)$ and $a\in\mathrm{TD}(\mathcal{B}(X,Y))$. Then $(a*g)_{-T}=a*g_{-T}\in\mathrm{TD}(Y)$.*

Proof. The proof of this result follows from a collection of observations. First of all, note that if $g \in \mathrm{TD}(X)$ and supp$g \in [T, \infty)$, it follows that $g_{-T} \in \mathrm{TD}(X)$. (See exercises.) Next,

$$\|\mathrm{G}(s)\| \leq C_{\mathrm{G}}(\mathrm{Re}\, s)|s|^{\nu} \qquad \forall s \in \mathbb{C}_+. \tag{3.5}$$

Since $g_{-T} \in \mathrm{CT}(X)$, then, by definition of the Laplace transform

$$\mathcal{L}\{g_{-T}\} = \langle g_{-T}, \exp(- \;\cdot\; s)\rangle = \langle g, \exp(-s(\;\cdot\; - T)\rangle = e^{sT}\mathrm{G}(s).$$

Since $g_{-T} \in \mathrm{TD}(X)$, then

$$\|e^{sT}\mathrm{G}(s)\| = e^{\mathrm{Re}\, s\, T}\|\mathrm{G}(s)\| \leq C_{\mathrm{G},T}(\mathrm{Re}\, s)|s|^{\nu'}. \tag{3.6}$$

In fact, comparing (3.5) and (3.6) it is clear that we can take $\nu = \nu'$ and the hypothesis on g_{-T} is just one on the behavior of $e^{\sigma T}C_{\mathrm{G}}(\sigma)$. (Note also that displacement does not modify the tempered character of a distribution and that the characterization of $\mathrm{TD}(X)$ as a space of derivatives of causal functions with polynomial growth leaves clear that what we are looking at is purely the preservation of causality by left displacement. More on this in Section 3.6.)

Therefore $\mathcal{L}\{a * g_{-T}\} = \mathrm{A}(s)\,(e^{sT}\mathrm{G}(s)) = e^{sT}\mathrm{A}(s)\mathrm{G}(s)$ and thus

$$\mathcal{L}\{(a * g_{-T})_T\} = e^{-sT}\mathcal{L}\{a * g_{-T}\} = \mathrm{A}(s)\mathrm{G}(s) \in \mathcal{A}(\mu + \nu, Y).$$

Finally, we have proved that $a * g_{-T} = (a * g)_{-T}$ and therefore $(a * g)_{-T}$ is causal as we wanted to prove. □

The following result –taken from [9]– gives a taste of the kind of estimates that can be obtained for convolution operators by working in the Laplace domain. It is based on a result by Christian Lubich [16], restricted to the class of symbols we are working with, and uses a couple of clever observations made by Lehel Banjai.

Proposition 3.2.2. *Let* $\mathrm{A} = \mathcal{L}\{a\} \in \mathcal{A}(\mu, \mathcal{B}(X, Y))$ *with* $\mu \geq 0$ *and let*

$$k := \lfloor \mu + 2 \rfloor \qquad \varepsilon := k - (\mu + 1) \in (0, 1].$$

If $g \in \mathcal{C}^{k-1}(\mathbb{R}, X)$ *is causal and* $g^{(k)}$ *is integrable, then* $a * g \in \mathcal{C}(\mathbb{R}, Y)$ *is causal and*

$$\|(a * g)(t)\| \leq 2^{\mu} C_{\varepsilon}(t) C_{\mathrm{A}}(t^{-1}) \int_0^t \|(\mathcal{P}_k g)(\tau)\| \mathrm{d}\tau, \tag{3.7}$$

where

$$C_{\varepsilon}(t) := \frac{1 + \varepsilon}{\pi\varepsilon}\,\frac{t^{\varepsilon}}{(1 + t)^{\varepsilon}}$$

and

$$(\mathcal{P}_k g)(t) := e^{-t}(e^{\cdot} g)^{(k)}(t) = \sum_{\ell=0}^{k} \binom{k}{\ell} g^{(\ell)}(t).$$

Proof. If $g^{(k)}$ is causal and integrable, it is tempered. Furthermore, it is the derivative of a continuous bounded causal function, and therefore $g^{(k)} \in \mathrm{TD}(X)$. Moreover, $\mathcal{L}\{\mathcal{P}_k g\} = (1+s)^k \mathrm{G}(s) \in \mathcal{A}(0, X)$ (see Proposition 3.1.3) and therefore $\mathrm{G} \in \mathcal{A}(-k, X)$ and

$$\|(1+s)^k \mathrm{G}(s)\| \le \int_0^\infty \|(\mathcal{P}_k g)(\tau)\| \mathrm{d}\tau \qquad \forall s \in \mathbb{C}_+. \tag{3.8}$$

A simple bound shows now that $\mathrm{A}\,\mathrm{G} \in \mathcal{A}(\mu - k, Y)$ and because $\mu - k = -(1+\varepsilon) < -1$, it follows from Proposition 3.1.1 that $a * g$ is continuous and causal. We can bound $(a*g)(t)$ using the strong form of the inversion formula, proceeding as follows:

$$\begin{aligned}
\|(a*g)(t)\| &\le \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^\infty \|\mathrm{A}(\sigma + \imath\omega)\mathrm{G}(\sigma+\imath\omega)\| \mathrm{d}\omega & &\text{(by (3.2))}\\
&\le \frac{e^{\sigma t}}{2\pi} C_{\mathrm{A}}(\sigma) \int_{-\infty}^\infty \|(\sigma+\imath\omega)^\mu \mathrm{G}(\sigma+\imath\omega)\| \mathrm{d}\omega & &(\mathrm{A} \in \mathcal{A}(\mu))\\
&\le \frac{e^{\sigma t}}{2\pi} C_{\mathrm{A}}(\sigma) \max_{\mathrm{Re}\, s = \sigma} \|(1+s)^k \mathrm{G}(s)\| \int_{-\infty}^\infty \frac{|\sigma+\imath\omega|^\mu}{|1+\sigma+\imath\omega|^k} \mathrm{d}\omega & &\\
&\le \frac{e^{\sigma t}}{\pi} C_{\mathrm{A}}(\sigma) \int_0^\infty \frac{|\sigma+\imath\omega|^\mu}{|1+\sigma+\imath\omega|^k} \mathrm{d}\omega \int_0^\infty \|(\mathcal{P}_k g)(\tau)\| \mathrm{d}\tau. & &\text{(by (3.8))}
\end{aligned}$$

We next bound

$$\begin{aligned}
C(\sigma) := \int_0^\infty \frac{|\sigma+\imath\omega|^\mu}{|1+\sigma+\imath\omega|^k} \mathrm{d}\omega \quad &\le \quad \int_0^\sigma \frac{(2\sigma)^\mu}{(1+\sigma)^k} \mathrm{d}\omega + \int_\sigma^\infty \frac{(2\omega)^\mu}{\omega^k} \mathrm{d}\omega\\
&= \quad (2\sigma)^\mu (1+\sigma)^{1-k} + \frac{2^\mu}{k - \mu - 1} \sigma^{1-k+\mu}\\
&= \quad 2^\mu \Big(\sigma^\mu (1+\sigma)^{1-k} + \varepsilon^{-1} \sigma^{-\varepsilon} \Big) \le 2^\mu (1 + \varepsilon^{-1}) \sigma^{-\varepsilon}.
\end{aligned}$$

Taking $\sigma = t^{-1}$ we obtain

$$\|(a*g)(t)\| \le 2^\mu C_\varepsilon(t) C_{\mathrm{A}}(t^{-1}) \int_0^\infty \|(\mathcal{P}_k g)(\tau)\| \mathrm{d}\tau,$$

which is a bound similar to (3.7) with $\|\mathcal{P}_k g\|$ integrated over $(0,\infty)$ instead of $(0,t)$. Let us fix $t > 0$ now and consider the function

$$p(\tau) := \begin{cases} g(\tau), & \tau \le t,\\ e^{-\tau} \displaystyle\sum_{\ell=0}^{k-1} \frac{(\tau - t)^\ell}{\ell!} (e^{\cdot} g)^{(\ell)}(t), & \tau \ge t. \end{cases}$$

Since p satisfies the same hypotheses as g, $a * p$ has the same properties as $a * g$. Also $p - g$ vanishes in $(-\infty, t)$ and by Proposition 3.2.1 the continuous function $a * (g - p)$ vanishes in $(-\infty, t)$ and therefore

$$\begin{aligned}
\|(a*g)(t)\| \quad &= \quad \|(a*p)(t)\| \le 2^\mu C_\varepsilon(t) C_{\mathrm{A}}(t^{-1}) \int_0^\infty \|(\mathcal{P}_k p)(\tau)\| \mathrm{d}\tau\\
&= \quad 2^\mu C_\varepsilon(t) C_{\mathrm{A}}(t^{-1}) \int_0^t \|(\mathcal{P}_k g)(\tau)\| \mathrm{d}\tau,
\end{aligned}$$

due to the fact that $\mathcal{P}_k p = 0$ in (t, ∞). □

Warning words. Passage through the Laplace domain is an elegant and powerful way of studying many convolution operators, but the direct and inverse transforms lose quite a lot of information. For instance, if $I : X \to X$ is the identity operator, then $\delta_0 \otimes I$ defines a distribution whose Laplace transform is the constant operator I, and it is therefore an element of the class $\mathcal{A}(0, \mathcal{B}(X,X))$. It is clear that Proposition 3.2.2 (that needs $k=2$) is not giving the best possible bound of that style (a sort of Sobolev embedding theorem in one variable). We have also seen an example (Section 2.6) of symbols $\mathrm{V}(s)$ and $\mathrm{V}^{-1}(s)$, reciprocally inverse, both of positive order.

3.3 Scattering by a sound-soft obstacle

The time domain single layer potential. By Proposition 2.6.3 and the preceding theory, there exists $\mathcal{S} \in \mathrm{TD}(\mathcal{B}(H^{-1/2}(\Gamma), H^1_\Delta(\mathbb{R}^d \setminus \Gamma)))$ such that $\mathcal{L}\{\mathcal{S}\} = \mathrm{S}$. Therefore, given $\lambda \in \mathrm{TD}(H^{-1/2}(\Gamma))$, the convolution

$$u = \mathcal{S} * \lambda$$

defines a causal tempered distribution with values in $H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$, that solves

$$\ddot{u} = \Delta u, \qquad [\![\gamma u]\!] = 0, \qquad [\![\partial_\nu u]\!] = \lambda. \tag{3.9}$$

(This can be seen by taking the Laplace transform $\mathrm{U}(s) = \mathrm{S}(s)\Lambda(s)$ and applying the properties of $\mathrm{S}(s)$.) Because equation (3.9) has at most a causal tempered solution (see Proposition 2.4.1), $\mathcal{S} * \lambda$ is characterized by the above transmission (and initial value) problem.

Remark. For this presentation of the theory it is not relevant to know what $\mathcal{S}$ is. What can be proved, using smooth densities λ (and a density argument), is that $\mathcal{S} * \lambda$ can be represented in integral form (see Chapter 1). The causal distribution $\mathcal{S}$ for $d = 3$ was found out in [12]: to $\varphi \in \mathcal{S}(\mathbb{R})$, it associates the operator

$$H^{-1/2}(\Gamma) \ni \lambda \longmapsto \int_\Gamma \frac{\varphi(\,\cdot\, - \mathbf{y})}{4\pi|\,\cdot\, - \mathbf{y}|}\,\lambda(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma).$$

The retarded single layer operator. If we define $\mathcal{V} := \gamma\mathcal{S}$ (γ is here a steady-state operator that is applied to the causal tempered distribution $\mathcal{S}$), it follows that $\mathcal{V} * \lambda \in \mathrm{TD}(H^{1/2}(\Gamma))$. We also know that $\mathcal{L}\{\mathcal{V}\} = \mathrm{V}$. An integral expression of $\mathcal{V} * \lambda$ can be obtained by taking the trace of $\mathcal{S} * \lambda$.

Extracting information from the bounds. The following bounds have been proved in Section 2.6:

$$\|\mathrm{S}(s)\|_{H^{-1/2}(\Gamma)\to H^1(\mathbb{R}^d\setminus\Gamma)} \le C\frac{|s|}{\sigma\underline{\sigma}^2} \qquad \|\mathrm{S}(s)\|_{H^{-1/2}(\Gamma)\to H^1_\Delta(\mathbb{R}^d\setminus\Gamma)} \le C\frac{|s|^2}{\sigma\underline{\sigma}^3}$$

$$\|\mathrm{V}(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C\frac{|s|}{\sigma\underline{\sigma}^2} \qquad \|\mathrm{V}^{-1}(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \le C\frac{|s|^2}{\sigma\underline{\sigma}}.$$

The first bound and Proposition 3.2.2 ($k = 3$, $\varepsilon = 1$) can be used to show that for λ smooth enough

$$\|(\mathcal{S} * \lambda)(t)\|_{1,\mathbb{R}^d} \le C\,\tfrac{t}{1+t}\,t\max\{1,t^2\}\int_0^t \|(\mathcal{P}_3\lambda)(\tau)\|_{-1/2,\Gamma}\mathrm{d}\tau.$$

Similarly (same Proposition, $\mu = 2$, $k = 4$ and $\varepsilon = 1$)

$$\|(\mathcal{V}^{-1} * g)(t)\|_{-1/2,\Gamma} \le C\,\tfrac{t}{1+t}\,t\max\{1,t\}\int_0^t \|(\mathcal{P}_4 g)(\tau)\|_{1/2,\Gamma}\mathrm{d}\tau.$$

These inequalities correspond to the two steps of solving the Dirichlet problem

$$\ddot{u} = \Delta u \qquad \gamma^+ u = \gamma^- u = g$$

with a boundary integral method: we first solve

$$\mathcal{V} * \lambda = g \qquad \Longleftrightarrow \qquad \lambda = \mathcal{V}^{-1} * g$$

and then input λ in the potential expression

$$u = \mathcal{S} * \lambda = \mathcal{S} * (\mathcal{V}^{-1} * g).$$

Associativity and more. The definition of convolution through the Laplace transform proves that associativity is satisfied:

$$\mathcal{S} * (\mathcal{V}^{-1} * g) = (\mathcal{S} * \mathcal{V}^{-1}) * g.$$

(Of course, the definition of $\mathcal{S}*\mathcal{V}^{-1}$ has to be done in a similar way.) To study the behavior of the convolution with the $\mathcal{B}(H^{1/2}(\Gamma), H^1(\mathbb{R}^d))$-valued causal distribution $\mathcal{S}*\mathcal{V}^{-1}$, we have to study the symbol $\mathrm{S}(s)\mathrm{V}^{-1}(s)$. In fact, we can prove that

$$\|\mathrm{S}(s)\mathrm{V}^{-1}(s)\|_{H^{1/2}(\Gamma)\to H^1(\mathbb{R}^d)} \le C\frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}}. \tag{3.10}$$

Proposition 3.2.2 ($\mu = 3/2$, $k = 3$, $\varepsilon = 1/2$) then yields:

$$\|u(t)\|_{1,\mathbb{R}^d} = \|(\mathcal{S} * \mathcal{V}^{-1} * g)(t)\|_{1,\mathbb{R}^d} \le C\,\tfrac{t^{1/2}}{1+t^{1/2}}\,t\max\{1,t^{3/2}\}\int_0^t \|(\mathcal{P}_3 g)(\tau)\|_{1/2,\Gamma}\mathrm{d}\tau.$$

Proof of (3.10) Given $\xi \in H^{1/2}(\Gamma)$, the function $u := \mathrm{S}(s)\mathrm{V}^{-1}(s)\lambda$ is the solution of the Dirichlet problem

$$\Delta u - s^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \qquad \gamma^\pm u = \xi.$$

To give a bound of the norm of this function we decompose it as $u = u_\xi + (u - u_\xi)$, where

$$\Delta u_\xi - |s|^2 u_\xi = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma \qquad \gamma u_\xi = \xi,$$

(this is the lifting of Proposition 2.5.1) and obviously $u - u_\xi \in H^1_0(\mathbb{R}^d \setminus \Gamma)$. Therefore

$$a_s(u, u - u_\xi) = 0. \tag{3.11}$$

From here on, the proof collects a simple coercivity argument and the lifting property Proposition 2.5.1. We first bound using coercivity and boundedness of the bilinear form a_s:

$$\begin{aligned}
\|\!|u - u_\xi|\!\|^2_{|s|,\mathbb{R}^d\setminus\Gamma} &\le \frac{|s|}{\sigma}|a_s(u - u_\xi, u - u_\xi)| && \text{(by (2.13))}\\
&= \frac{|s|}{\sigma}|a_s(-u_\xi, u - u_\xi)| && \text{(by (3.11))}\\
&\le \frac{|s|}{\sigma}\|\!|u_\xi|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma}\|\!|u - u_\xi|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma}. && \text{(by (2.12))}
\end{aligned}$$

Next

$$\begin{aligned}
\|\!|u|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma} &\le \|\!|u - u_\xi|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma} + \|\!|u_\xi|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma}\\
&\le \frac{2|s|}{\sigma}\|\!|u_\xi|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma} && \text{(see above)}\\
&\le \frac{2|s|}{\sigma}C_\Gamma \max\{1, |s|\}^{1/2}\|\xi\|_{1/2,\Gamma} && \text{(by Proposition 2.5.1)}\\
&\le 2C_\Gamma\frac{|s|^{3/2}}{\sigma\underline{\sigma}^{1/2}}\|\xi\|_{1/2,\Gamma},
\end{aligned}$$

which concludes the proof since $\underline{\sigma}\|u\|_{1,\mathbb{R}^d\setminus\Gamma} \le \|\!|u|\!\|_{|s|,\mathbb{R}^d\setminus\Gamma}$.

Concerning $\mathcal{V}^{-1}$. Just some warning words about the notation $\mathcal{V}^{-1}$. In the Laplace domain,

$$\mathrm{V}^{-1}(s)\mathrm{V}(s) = I \qquad \forall s \in \mathbb{C}_+,$$

where $I : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the identity operator. When we take the inverse Laplace transform, we obtain

$$\mathcal{V}^{-1} * \mathcal{V} = \delta_0 \otimes I,$$

that is, the tempered distribution $\mathcal{V}^{-1}$ is a convolutional inverse of $\mathcal{V}$, which somehow justifies this notation. It is not an inverse of $\mathcal{V} : \mathcal{S}(\mathbb{R}) \to \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ though.

3.4 The double layer potential

In the Laplace domain, the double layer potential is defined with the integral form

$$H^{1/2}(\Gamma) \ni \xi \longmapsto \mathrm{D}(s)\xi := \int_\Gamma \nabla_{\mathbf{y}}E(\,\cdot\,, \mathbf{y}; s)\cdot\boldsymbol{\nu}(\mathbf{y})\,\xi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}). \tag{3.12}$$

Here E is the fundamental solution of $\Delta - s^2$ (recall (2.10)). A simple computation shows that

$$\nabla_{\mathbf{y}}E(\mathbf{x}, \mathbf{y}; s)\cdot\boldsymbol{\nu}(\mathbf{y}) = \begin{cases} -\dfrac{s}{4}H^{(1)}_1(\imath s|\mathbf{x}-\mathbf{y}|)\dfrac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} & (d=2)\\ \dfrac{e^{-s|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\dfrac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\Big(s + \dfrac{1}{|\mathbf{x}-\mathbf{y}|}\Big) & (d=3).\end{cases}$$

For $s \in \mathbb{C}_+$, the double layer potential provides the unique solution $u = \mathrm{D}(s)\xi \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ to the transmission problem:

$$\Delta u - s^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \qquad [\![\gamma u]\!] = -\xi, \qquad [\![\partial_\nu u]\!] = 0. \tag{3.13}$$

The integral operators associated to $\mathrm{D}(s)$ are

$$\mathrm{K}(s) := \{\!\{\gamma \mathrm{D}(s)\}\!\} \qquad \mathrm{W}(s) := \partial_\nu^\pm \mathrm{D}(s).$$

We next collect some Laplace domain bounds related to these operators [3, 13]. The techniques are very similar to those of the single layer operator (Section 2.6), and the reader is encouraged to move on and try to find what are the associated time domain bounds.

Proposition 3.4.1. *There exist generic constants (all denoted C_Γ) such that for all $s \in \mathbb{C}_+$:*

$$\mathrm{Re}\Big(e^{-\imath \mathrm{Arg}\, s}\langle \mathrm{W}(s)\xi, \overline{\xi}\rangle_\Gamma\Big) \ge C_\Gamma \frac{\sigma \underline{\sigma}^2}{|s|}\|\xi\|^2_{1/2,\Gamma} \qquad \forall \xi \in H^{1/2}(\Gamma), \tag{3.14}$$

$$\|\mathrm{W}(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \le C_\Gamma \frac{|s|^2}{\sigma\underline{\sigma}} \tag{3.15}$$

$$\|\mathrm{W}^{-1}(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C_\Gamma \frac{|s|}{\sigma\underline{\sigma}^2} \tag{3.16}$$

$$\|\mathrm{D}(s)\|_{H^{1/2}(\Gamma)\to H^{1}(\mathbb{R}^d\setminus\Gamma)} \le C_\Gamma \frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}} \tag{3.17}$$

$$\|\mathrm{D}(s)\|_{H^{1/2}(\Gamma)\to H^{1}_\Delta(\mathbb{R}^d\setminus\Gamma)} \le C_\Gamma \frac{|s|^{5/2}}{\sigma\underline{\sigma}^{5/2}}. \tag{3.18}$$

Proof. Let $\xi \in H^{1/2}(\Gamma)$ and $u := \mathrm{D}(s)\xi$. Since $[\![\gamma u]\!] = -\xi$ and $\partial_\nu^\pm u = -\mathrm{W}(s)\xi$, the definition of the weak normal derivatives yields

$$\langle \mathrm{W}(s)\xi, \overline{\xi}\rangle_\Gamma = \langle -\partial_\nu u, -[\![\gamma \overline{u}]\!]\rangle_\Gamma = \langle \partial_\nu^- u, \gamma^- \overline{u}\rangle_\Gamma - \langle \partial_\nu^+ u, \gamma^+ \overline{u}\rangle_\Gamma = a_s(u, \overline{u}) \tag{3.19}$$

and therefore

$$\begin{aligned}
\mathrm{Re}\left(e^{-\imath \mathrm{Arg}\, s}\langle \mathrm{W}(s)\xi, \overline{\xi}\rangle_\Gamma\right) &= \frac{\sigma}{|s|}|\!|\!| u |\!|\!|^2_{|s|,\mathbb{R}^d\setminus\Gamma} && \text{(coercivity of } a_s\text{: (2.13))}\\
&\ge \frac{\sigma\underline{\sigma}^2}{|s|}\|u\|^2_{1,\mathbb{R}^d\setminus\Gamma} && \text{(by (2.14))}\\
&\ge \|\gamma\|^{-2}\frac{\sigma\underline{\sigma}^2}{|s|}\|\xi\|_{1/2,\Gamma}, && \text{(trace theorem)}
\end{aligned}$$

which proves (3.14) and therefore (3.16). Going back to (3.19) and using the bound for the normal derivative given in Proposition 2.5.2, it follows that

$$|\!|\!| u |\!|\!|^2_{|s|,\mathbb{R}^d\setminus\Gamma} \le \frac{|s|}{\sigma}\|\partial_\nu u\|_{-1/2,\Gamma}\|\xi\|_{1/2,\Gamma} \le C_\Gamma \frac{|s|^{3/2}}{\sigma\underline{\sigma}^{1/2}}|\!|\!| u |\!|\!|_{|s|,\mathbb{R}^d\setminus\Gamma}\|\xi\|_{1/2,\Gamma}$$

and therefore, since

$$\|u\|_{1,\mathbb{R}^d\setminus\Gamma} \le \frac{1}{\underline{\sigma}} |\!|\!| u |\!|\!|_{|s|,\mathbb{R}^d\setminus\Gamma} \qquad \|\Delta u\|_{\mathbb{R}^d\setminus\Gamma} = |s| \|su\|_{\mathbb{R}^d\setminus\Gamma} \le |s| |\!|\!| u |\!|\!|_{|s|,\mathbb{R}^d\setminus\Gamma},$$

and

$$\|\mathrm{W}(s)\xi\|_{-1/2,\Gamma} = \|\partial_\nu u\|_{-1/2,\Gamma} \le C_\Gamma \frac{|s|^{1/2}}{\underline{\sigma}^{1/2}} |\!|\!| u |\!|\!|_{|s|,\mathbb{R}^d\setminus\Gamma}$$

the proof of (3.15), (3.17), and (3.18) follows readily. □

As a direct consequence of the general theory we have established for our class of symbols in the Laplace domain, we can define time domain operator valued distributions

$$\begin{array}{rcl} \mathcal{D} & \in & \mathrm{TD}(\mathcal{B}(H^{1/2}(\Gamma), H^1_\Delta(\mathbb{R}^d \setminus \Gamma))), \\ \mathcal{K} & \in & \mathrm{TD}(\mathcal{B}(H^{1/2}(\Gamma), H^{1/2}(\Gamma))), \\ \mathcal{W} & \in & \mathrm{TD}(\mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))), \\ \mathcal{W}^{-1} & \in & \mathrm{TD}(\mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))). \end{array}$$

Note that by definition (recall (3.13)) if $\xi \in \mathrm{TD}(H^{1/2}(\Gamma)$, then $u := \mathcal{D} * \xi$ is the unique (causal tempered) solution of

$$\ddot{u} = \Delta u, \qquad [\![\gamma u]\!] = -\xi, \qquad [\![\partial_\nu u]\!] = 0. \tag{3.20}$$

3.5 Kirchhoff's formula

The previous results leave us in situation to derive the entire Calderón (boundary integral) calculus for the wave equation. It all starts with the weak version of Kirchhoff's formula:

Proposition 3.5.1. *Causal solutions of the wave equation*

$$u \in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma) \qquad \ddot{u} = \Delta u \tag{3.21}$$

(equality as casual tempered distributions with values in $L^2(\mathbb{R}^d \setminus \Gamma)$) can be represented as

$$u = \mathcal{S} * [\![\partial_\nu u]\!] - \mathcal{D} * [\![\gamma u]\!]. \tag{3.22}$$

Proof. As already mentioned in Chapter 1, the keys to this representation are a uniqueness theorem for transmission problems (that is the role played by Proposition 2.4.1) and the interpretation of layer potentials as solutions of transmission problems, i.e., (3.9) for the single layer potential and (3.20) for the double layer potential. Given u satisfying (3.21) we can define

$$\varphi := [\![\gamma u]\!] \in \mathrm{TD}(H^{1/2}(\Gamma)) \quad \text{and} \quad \lambda := [\![\partial_\nu u]\!] \in \mathrm{TD}(H^{-1/2}(\Gamma))$$

and note that the distribution

$$v := u - \mathcal{S} * \lambda + \mathcal{D} * \varphi \in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$$

is a causal solution of

$$\ddot{v} = \Delta u \qquad [\![\gamma v]\!] = 0 \qquad [\![\partial_\nu v]\!] = 0$$

and therefore $v = 0$. □

Several useful formulas follows from the representation (3.22) and the definition of the integral operators. For instance, for any solution of (3.21) we can write

$$\begin{bmatrix} \{\!\!\{\gamma u\}\!\!\} \\ \{\!\!\{\partial_\nu u\}\!\!\} \end{bmatrix} = \begin{bmatrix} \mathcal{V} & -\mathcal{K} \\ \mathcal{K}^t & \mathcal{W} \end{bmatrix} * \begin{bmatrix} [\![\partial_\nu u]\!] \\ [\![\gamma u]\!] \end{bmatrix}.$$

	$[\![\gamma\,\cdot\,]\!]$	$[\![\partial_\nu\,\cdot\,]\!]$
$\mathcal{S}*\lambda$	0	λ
$\mathcal{D}*\varphi$	$-\varphi$	0

	$\{\!\!\{\gamma\,\cdot\,\}\!\!\}$	$\{\!\!\{\partial_\nu\,\cdot\,\}\!\!\}$
$\mathcal{S}*\lambda$	$\mathcal{V}*\lambda$	$\mathcal{K}^t*\lambda$
$\mathcal{D}*\varphi$	$\mathcal{K}*\varphi$	$-\mathcal{W}*\varphi$

Table 3.1: The jump relations in two tables.

3.6 Another look at causality

There are several aspects to causality in the wave equation that are more or less hidden in the Laplace transform. We have already seen that if data are delayed (their support starts at time $T > 0$), so is the solution. (This was Proposition 3.2.1, which was just a simple statement on the convolutional character of all our operators.) We next see how finite speed of propagation is also observable in the Laplace transform. It all follows from the following simple abstract result (which was also hidden in the proof of Proposition 3.2.1):

Proposition 3.6.1. *Let* $\mathrm{F} \in \mathcal{A}(\mu, X)$, $\mathrm{F} = \mathcal{L}\{f\}$ *and*

$$(0,\infty) \ni \sigma \longmapsto C(\sigma) := \sup_{\mathrm{Re}\, s \ge \sigma} \|s^{-\mu}\mathrm{F}(s)\|.$$

If for some $T > 0$, *the function* $C(\sigma)e^{\sigma T}$ *is non-increasing, then* f_{-T} *is causal.*

Proof. Note that the function $C(\sigma)$ is well defined, non-increasing and admits a rational bound as $\sigma \to 0$. (This is just a simple consequence of the fact that $C \le C_{\mathrm{F}}$.) If we define $G(s) := e^{sT}\mathrm{G}(s)$, the above hypotheses imply that $\mathrm{G} \in \mathcal{A}(\mu, X)$. Let then $g \in \mathcal{L}^{-1}\{\mathrm{G}\} \in \mathrm{TD}(X)$. It is simple to see that

$$\mathcal{L}\{g_T\} = e^{-sT}\mathrm{G}(s) = \mathrm{F}(s),$$

which means that $g_T = f$ and therefore $f_{-T} = g$ is causal. $\square$

Another way of checking the hypothesis of Proposition 3.6.1 is the following: if we can write

$$\|\mathrm{F}(s)\| \le e^{-T\,\mathrm{Re}\,s}\, D_{\mathrm{F}}(\mathrm{Re}\,s)\, |s|^\mu \qquad \forall s \in \mathbb{C}_+$$

for some $T > 0$, where D_{F} is non-increasing and $D_{\mathrm{F}}(\sigma) \le C\sigma^{-m}$ for $\sigma \in (0,1)$, then, f_{-T} is causal, that is, the temporal support of f is contained in $[T, \infty)$.

Given the intimate relationship between causality and finite speed of propagation, it is not surprising that this delayed causality property shows how distance from the scatterer Γ gives a delay in the arrival of the solution.

Proposition 3.6.2. *Let $B \subset \mathbb{R}^d$ be such that $\overline{B} \cap \Gamma = \emptyset$ and let*

$$T := \mathrm{dist}(\overline{B}, \Gamma) := \inf\{|\mathbf{x} - \mathbf{y}| \,:\, \mathbf{x} \in B,\, \mathbf{y} \in \Gamma\} > 0.$$

If

$$u \in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \qquad \ddot{u} = \Delta u,$$

then $u_{-T}|_B$ is causal, that is, the temporal support of the distribution $u|_B$ is contained in $[T, \infty)$.

Proof. Thanks to Kirchhoff's formula, we only need to check that this delay property is satisfied by single and double layer potentials. We will only give full details for the single layer potential in three dimensions. The arguments in two dimensions are very similar. The proof of the double layer potential is even simpler.

We are going to show that

$$\sup_{\mathbf{x}\in B} \left\| \frac{e^{-s|\mathbf{x}-\,\cdot\,|}}{4\pi|\mathbf{x}-\,\cdot\,|} \right\|_{1/2,\Gamma} \le e^{-\sigma T} \alpha(T) \frac{|s|^{1/2}}{\underline{\sigma}^{1/2}}, \qquad \sigma = \mathrm{Re}\, s. \tag{3.23}$$

Since for $\mathbf{x} \not\in \Gamma$, the layer potential $\mathrm{S}(s)$ admits the form

$$(\mathrm{S}(s)\lambda)(\mathbf{x}) = \langle \lambda, \frac{e^{-s|\mathbf{x}-\,\cdot\,|}}{4\pi|\mathbf{x}-\,\cdot\,|} \rangle_\Gamma,$$

($\mathrm{S}(s)\lambda$ is actually a $\mathcal{C}^\infty$ function in $\mathbb{R}^d \setminus \Gamma$), the bound (3.23) gives

$$\|\mathrm{S}(s)\lambda\|_{H^{-1/2}(\Gamma)\to L^2(B)} \le C\, e^{-\sigma T} \alpha(T)\, \frac{|s|^{1/2}}{\underline{\sigma}^{1/2}}$$

and therefore Proposition 3.6.1 shows that $\mathcal{S} * \lambda$, restricted to B, (this means that we compose with the embedding operator $H^1(\mathbb{R}^d \setminus \Gamma) \to L^2(B)$) is supported in $[T, \infty)$.

To show (3.23) let us first denote

$$f := \frac{e^{-s|\mathbf{x}-\,\cdot\,|}}{|\mathbf{x}-\,\cdot\,|} \,:\, \Gamma \to \mathbb{C}.$$

Note that

$$\|f\|_\Gamma \le |\Gamma|^{1/2} \|f\|_{L^\infty(\Gamma)} \le \frac{|\Gamma|^{1/2}}{T} e^{-\sigma T} \tag{3.24}$$

and that

$$\|f\|^2_{1,\Gamma} = \|f\|^2_\Gamma + \|\nabla_\Gamma f\|^2_\Gamma \le |\Gamma| \left(\|f\|^2_{L^\infty(\Gamma)} + \|\nabla f\|^2_{L^\infty(\Gamma)} \right) \tag{3.25}$$

(the first gradient is tangential and the second one is not). Since

$$\nabla f(\mathbf{y}) = \frac{e^{-s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \left(s + \frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|},$$

it is clear that

$$\|\nabla f\|_{L^\infty(\Gamma)} \le \frac{e^{-\sigma T}}{T} \left(|s| + \frac{1}{T} \right). \tag{3.26}$$

Plugging (3.24) and (3.26) in (3.25) and bounding $|s| + \eta \leq 2|s| \max\{1, \eta\}/\underline{\sigma}$, we can bound

$$\|f\|_{1,\Gamma} \leq |\Gamma|^{1/2} \frac{e^{-\sigma T}}{T} \left(1 + \frac{1}{T}\right) \frac{|s|}{\underline{\sigma}}$$

and finally using an interpolation property of Sobolev space

$$\|f\|_{1/2,\Gamma} \leq \|f\|_{\Gamma}^{1/2} \|f\|_{1,\Gamma}^{1/2}$$

we prove (3.23). □

Incident waves. Similar bounds can be used to prove that the type of incident waves (spherical and plane waves) can be used to provide data for scattering problems with very weak conditions on the transmitted signal.

3.7 Coercivity in the time domain

The two coercivity properties we have obtained in the Laplace domain, namely (Proposition 2.6.1)

$$\mathrm{Re}\left(e^{\imath \mathrm{Arg}\, s} \langle \overline{\lambda}, \mathrm{V}(s)\lambda \rangle_\Gamma\right) \geq C_\Gamma \frac{\sigma \underline{\sigma}}{|s|^2} \|\lambda\|_{-1/2,\Gamma}^2 \qquad \forall \lambda \in H^{-1/2}(\Gamma). \tag{3.27}$$

and (Proposition 3.4.1)

$$\mathrm{Re}\Big(e^{-\imath \mathrm{Arg}\, s} \langle \mathrm{W}(s)\xi, \overline{\xi} \rangle_\Gamma\Big) \geq C_\Gamma \frac{\sigma \underline{\sigma}^2}{|s|} \|\xi\|_{1/2,\Gamma}^2 \qquad \forall \xi \in H^{1/2}(\Gamma),$$

can be rewritten as some kind of coercivity properties the associated boundary integral operators in the time domain.

Plancherel's Formula. Consider a real Hilbert space X, whose inner product is denoted $\langle \,\cdot\,, \,\cdot\, \rangle_X$. Associated to X, there is a complex Hilbert space $X + \imath X$: its corresponding inner product can be constructed by separating real and imaginary parts. (We will keep the bracket bilinear instead of sesquilinear though.) Given $f \in L^2(\mathbb{R}, X)$, its Fourier transform $\mathcal{F}\{f\} \in L^2(\mathbb{R}, X + \imath X)$ is an extension of the operator (defined, for instance, for compactly supported functions):

$$\mathcal{F}\{f\}(\eta) = \int_{-\infty}^{\infty} e^{-\imath \eta t} f(t) \mathrm{d}t$$

(the integral is in the sense of Bochner in X). Plancherel's Formula is the statement

$$\int_{-\infty}^{\infty} \langle \overline{\mathcal{F}\{f\}(\eta)}, \mathcal{F}\{g\}(\eta) \rangle_X \mathrm{d}\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f(t), g(t) \rangle_X \mathrm{d}t,$$

which is actually in the basis of the possibility of defining $\mathcal{F}$ for functions in $L^2(\mathbb{R}, X)$. When X is the pivotal space of a Gelfand triple

$$Y \subset X \cong X' \subset Y'$$

(like $H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$), the result can be extended for $f \in L^2(\mathbb{R}, Y')$ and $g \in L^2(\mathbb{R}, Y)$.

Laplace and Fourier. If $f \in L^2((0,\infty),X)$ (extended by zero for negative t), we can easily write

$$\mathrm{F}(s) = \int_0^\infty e^{-s\,t} f(t)\mathrm{d}t = \mathcal{F}\{e^{-\sigma t} f(t)\}(\eta) \qquad s = \sigma + \imath\eta.$$

Therefore if $\mathrm{A} \in \mathcal{A}(\mu', Y')$ and $\mathrm{B} \in \mathcal{A}(\mu, Y)$, with $\mu + \mu' < -1$, we can integrate the duality

$$\langle \overline{\mathrm{A}(s)}, \mathrm{B}(s)\rangle_X$$

along a line $\mathrm{Re}\, s = \sigma$ (the fact that $\mu + \mu' < -1$ allows us to do that), and obtain

$$\int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \langle \overline{\mathrm{A}(s)}, \mathrm{B}(s)\rangle_X \mathrm{d}s = \frac{1}{2\pi}\int_0^\infty e^{-2\sigma t}\langle a(t), b(t)\rangle_X \mathrm{d}t.$$

Coercivity for the single layer operator. We can then integrate the following expression, equivalent to (3.27),

$$\begin{aligned} \mathrm{Re}\left(\langle \overline{\Lambda(s)}, \mathrm{V}(s)s\Lambda(s)\rangle_\Gamma\right) &\geq C_\Gamma \sigma \underline{\sigma} \|s^{-1/2}\Lambda(s)\|^2_{-1/2,\Gamma} \\ &= C_\Gamma \sigma \underline{\sigma} (\overline{s^{-1/2}\Lambda(s)}, s^{-1/2}\Lambda(s))_{-1/2,\Gamma}. \end{aligned} \tag{3.28}$$

(The bracket in the right hand side is the $H^{-1/2}(\Gamma)$-inner product.) Formally, we obtain

$$\int_0^\infty e^{-2\sigma\,t}\langle \lambda(t), (\mathcal{V} * \dot\lambda)(t)\rangle_\Gamma \mathrm{d}t \geq C_\Gamma \sigma\underline{\sigma} \int_0^\infty e^{-2\sigma t}\|\partial^{-1/2}\lambda(t)\|^2_{-1/2,\Gamma}\mathrm{d}t, \tag{3.29}$$

where $\partial^{-1/2}\lambda = \mathcal{L}^{-1}\{s^{-1/2}\Lambda(s)\}$.

We can keep track of the symbols, starting with

$$\Lambda \in \mathcal{A}(\mu, H^{-1/2}(\Gamma))$$

so that

$$s\mathrm{V}(s)\Lambda(s) \in \mathcal{A}(\mu+2, H^{1/2}(\Gamma)) \qquad s^{-1/2}\Lambda(s) \in \mathcal{A}(\mu - \tfrac{1}{2}, H^{-1/2}(\Gamma)).$$

The requirements for integrating are different in the left and right hand sides of (3.28). For the left hand side we need $2\mu + 2 < -1$, that is, $\mu < -3/2$. For the right hand side, it is enough with $2\mu - 1 < -1$ ($\mu < 0$). This mismatch in time regularity was already noticed in the seminal paper of Bamberger and Ha-Duong. In that work, some anisotropic Sobolev spaces are introduced by freezing a particular value of σ. The kind of densities they deal with are those causal distributions (that admit Laplace transform) such that $e^{-\sigma\,\cdot}\,\dot\lambda \in L^2((0,\infty), H^{-1/2}(\Gamma))$ and $e^{-\sigma\,\cdot}\,\mathcal{V} * \dot\lambda \in L^2((0,\infty), H^{1/2}(\Gamma))$. With that Hilbert space structure, (3.29) can be used as the starting point of a stability analysis for Galerkin methods for equations

$$\mathcal{V} * \lambda = g,$$

based on the weak formulation

$$\int_0^\infty e^{-2\sigma t}\langle \mu(t), (\mathcal{V} * \dot\lambda)(t)\rangle_\Gamma \mathrm{d}t = \int_0^\infty e^{-2\sigma t}\langle \mu, \dot g(t)\rangle_\Gamma \mathrm{d}t.$$

Note that the type of Galerkin methods that we introduced in Section 1.7 did not include the exponential factor. To the best of my knowledge, there is no theoretical support for the elimination of this weight but experiments show that it is not needed.

3.8 Exercises

1. (Section 3.1) In the following exercises we fill in the gaps of the proof of Proposition 3.1.1. Let $\mathrm{F} : \mathbb{C}_+ \to X$ satisfy

$$\|\mathrm{F}(s)\| \le C_{\mathrm{F}}(\mathrm{Re}\, s)|s|^{\mu} \qquad \forall s \in \mathbb{C}_+$$

where $C_{\mathrm{F}} : (0,\infty) \to (0,\infty)$ is non-increasing, and consider the functions (for $\sigma > 0$)

$$f_\sigma(t) := \frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{st}\mathrm{F}(s)\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sigma+\imath\omega}\mathrm{F}(\sigma+\imath\omega)\mathrm{d}\omega.$$

   (a) Show that f_σ is well defined and

$$\|f_\sigma(t)\| \le \frac{1}{2\pi}C_{\mathrm{F}}(\sigma)\sigma^{1+\mu}e^{\sigma t}\int_{-\infty}^{\infty}(1+\omega^2)^{\mu/2}\mathrm{d}\omega.$$

   (b) Use the Dominated Convergence Theorem to show that f_σ is a continuous function of t.

   (c) For fixed $0 < \sigma_1 < \sigma_2$ and $t \in \mathbb{R}$ we consider the quantities

$$\varepsilon_R^{\pm} := \int_{\sigma_1 \pm \imath R}^{\sigma_2 \pm \imath R} e^{st}\mathrm{F}(s)\mathrm{d}s = e^{\pm R\imath t}\int_{\sigma_1}^{\sigma_2} e^{\omega t}\mathrm{F}(\omega \pm \imath R)\mathrm{d}\omega.$$

   Show that $\lim_{R\to\infty}\varepsilon_R = 0$. Use this and Cauchy's Homology Theorem to show that $f_{\sigma_1}(t) = f_{\sigma_2}(t)$. We can then write $f(t) := f_\sigma(t)$.

   (d) Use the previous result and the bound of (a) to show that when $t < 0$, $f(t) = 0$.

   (e) Show that if $C_{\mathrm{F}}(1/\sigma)$ has tempered growth as $\sigma \to \infty$, then $f : \mathbb{R} \to X$ defines a causal tempered distribution. (**Hint.** As suggested in Section 3.1, take $\sigma = 1/t$ in the bound of (a).)

2. (Section 3.1) Assume that $\mathrm{F} : \mathbb{C}_+ \to X$ satisfies

$$\|\mathrm{F}(s)\| \le C_{\mathrm{F}}(\mathrm{Re}\, s)|s|^{\mu} \qquad \forall s \in \mathbb{C}_+$$

where $C_{\mathrm{F}} : (0,\infty) \to (0,\infty)$ is non-increasing. Show that

$$\frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty}\frac{1}{z-s}\mathrm{F}(s)\mathrm{d}s = \begin{cases} 0 & 0 < \mathrm{Re}\, z < \sigma \\ \mathrm{F}(z) & 0 < \sigma < \mathrm{Re}\, z, \end{cases}$$

You can proceed as follows:

   (a) If $\mathrm{Re}\, z < \sigma$, show that the integral

$$\frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty}\frac{1}{z-s}\mathrm{F}(s)\mathrm{d}s$$

   does not depend on σ.

(b) Using (a) and taking $\sigma \to \infty$, the first part of the result is proved.

(c) When $0 < \sigma < \operatorname{Re} z$, take $\sigma' > \operatorname{Re} z$, use Cauchy's Representation Formula in a closed rectangular contour with vertices $\sigma \pm \imath R$ and $\sigma' \pm \imath R$ (with R large enough so that z is enclosed by the contour) and a limiting argument to show that
$$\frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty}\frac{1}{z-s}\mathrm{F}(s)\mathrm{d}s = F(z) + \frac{1}{2\pi\imath}\int_{\sigma'-\imath\infty}^{\sigma'+\imath\infty}\frac{1}{z-s}\mathrm{F}(s)\mathrm{d}s.$$
Then use (b) to prove the result.

3. (Section 3.1) Let F satisfy the same hypotheses and in the previous two problems. Show then that the Laplace transform of
$$f(t) := \frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{st}\mathrm{F}(s)\mathrm{d}s.$$
is F. (**Hint.** Use Problem 2.)

4. (Section 3.2) Show that if $g \in \mathrm{TD}(X)$ and $\operatorname{supp} g \subset [T,\infty)$, then $g_{-T} \in \mathrm{TD}(X)$. This can be done in the time domain, by writing $g = \phi^{(k)}$, where $\phi : \mathbb{R} \to X$ is a continuous causal function with polynomial growth (Proposition 3.1.3) and then showing that $\operatorname{supp}\phi \subset [T,\infty)$ (see Exercises in the previous chapter).

5. (Section 3.4) Using Propositions 3.4.1 and 3.2.2, give bounds for
$$\|(\mathcal{D} * \xi)(t)\|_{1,\mathbb{R}^d}, \qquad \|(\mathcal{W} * \xi)(t)\|_{-1/2,\Gamma}, \quad \text{and} \quad \|(\mathcal{W}^{-1} * h)(t)\|_{1/2,\Gamma}.$$

6. (Section 3.4). Find a bound of $\|\mathrm{D}(s)\mathrm{W}^{-1}(s)\|_{H^{-1/2}(\Gamma)\to H^1(\mathbb{R}^d\setminus\Gamma)}$ and use to find bounds for $\|(\mathcal{D} * \mathcal{W}^{-1} * h)(t)\|_{1,\mathbb{R}^d\setminus\Gamma}$. As a hint, prove that if $u = \mathrm{D}(s)\mathrm{W}^{-1}(s)h$ where $h \in H^{-1/2}(\Gamma)$, then
$$a_{s,\mathbb{R}^d}(u,v) = \langle h, \gamma^+ v - \gamma^- v\rangle_\Gamma \quad \forall v \in H^1(\mathbb{R}^d \setminus \Gamma).$$

7. (Sections 3.3, 3.4, and 3.5) The exterior Dirichlet to Neumann operator $\mathrm{DtN}^+(s) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined by
$$\mathrm{DtN}^+(s)\xi := \partial_\nu u \qquad \text{where} \qquad \left[\begin{array}{ll} \Delta u - s^2 u = 0 & \text{in } \Omega_+, \\ \gamma^+ u = \xi. & \end{array}\right.$$
Its inverse is the Neumann to Dirichlet operator $\mathrm{NtD}^+(s)$.

(a) Prove the following coercivity estimates
$$-\operatorname{Re}\left(e^{-\imath \operatorname{Arg} s}\langle \mathrm{DtN}^+(s)\xi, \overline{\xi}\rangle_\Gamma\right) \ge C_\Gamma \frac{\sigma\underline{\sigma}^2}{|s|^2}\|\xi\|^2_{1/2,\Gamma} \qquad \forall \xi \in H^{1/2}(\Gamma), \qquad \forall s \in \mathbb{C}_+,$$
$$-\operatorname{Re}\left(e^{\imath \operatorname{Arg} s}\langle \overline{\lambda}, \mathrm{NtD}^+(s)\lambda\rangle_\Gamma\right) \ge C_\Gamma \frac{\sigma\underline{\sigma}}{|s|^2}\|\lambda\|^2_{1/2,\Gamma} \qquad \forall \lambda \in H^{-1/2}(\Gamma), \qquad \forall s \in \mathbb{C}_+.$$

(b) Use the previous estimates to give bounds of

$$\|\mathrm{DtN}^+(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \qquad \text{and} \qquad \|\mathrm{NtD}^+(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)}.$$

(c) Write down the corresponding estimates for the time domain Dirichlet to Neumann and Neumann to Dirichlet operators.

(d) Show that

$$\mathrm{DtN}^+(s) = (-\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t(s))\mathrm{V}^{-1}(s) = \mathrm{V}^{-1}(s)(-\tfrac{1}{2}\mathrm{I} + \mathrm{K}(s))$$

(**Hint.** For the first identity use a single layer potential representation. For the second one, use a direct boundary integral representation of the exterior solution.)

8. (Section 3.6) Complete the proof of Proposition 3.6.2 by providing estimates for the double layer potential in three dimensions and for the single and double layer potentials in two dimensions. Note that in the two dimensional case you will need to use estimates for the behavior of the Hankel functions $H_0^{(1)}$ and $H_1^{(1)}$.

9. (Section 3.7) Using the coercivity estimate of the form

$$\mathrm{Re}\,\langle \mathrm{W}(s)\xi, \overline{s\xi}\rangle_\Gamma \geq C_\Gamma \sigma \underline{\sigma}^2 \|\xi\|^2_{1/2,\Gamma} \qquad \forall \xi \in H^{1/2}(\Gamma),$$

derive a formal coercivity estimate in the time domain for the convolution operator $\xi \mapsto \mathcal{W} * \xi$. Write down what the corresponding variational formulation for the problem $\mathcal{W} * \xi = h$ would be.

Chapter 4

Convolution Quadrature

Convolution quadrature is a discretization technique for causal convolutions and convolution equations. Coming from a mathematical argument that might seem bizarre at the beginning, this method ends up using data in the time domain but the Laplace transform of the operator. This mixture of Laplace and time domain looks somewhat unnatural but yields a general type of methods that can be easily used in black-box fashion. The method and much of its initial development are due to Christian Lubich.

4.1 Discrete convolutions with CQ

The aim of this first section is to introduce in a simple and non-rigorous way one of the convolution quadrature methods, based upon the backwards Euler formula. Let $f : [0,\infty) \to \mathcal{B}(X,Y)$ be a (causal) operator-valued function for which the Laplace transform

$$\mathcal{L}\{f\}(s) = \mathrm{F}(s) := \int_0^\infty e^{-s\,t} f(t)\mathrm{d}t$$

exists. Although this is not important at this point of our exposition, let us assume that $\mathrm{F}(s)$ exists for all $s \in \mathbb{C}_+$ and that it decays fast enough so that the inversion formula

$$f(t) = \frac{1}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{s\,t}\mathrm{F}(s)\mathrm{d}s$$

holds for all $\sigma > 0$ (see Section 3.1). Let now $g : [0,\infty) \to X$. We first aim to approximate the convolution $f * g : [0,\infty) \to Y$

$$(f * g)(t) := \int_0^t f(\tau)\, g(t-\tau)\mathrm{d}\tau.$$

At least formally (we are not going to verify conditions for Fubini's theorem yet)

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \\
&= \int_0^t \left(\frac{1}{2\pi\,\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{s\tau}\,\mathrm{F}(s)\mathrm{d}s \right) g(t-\tau)\mathrm{d}\tau \\
&= \frac{1}{2\pi\,\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \mathrm{F}(s) \left(\int_0^t e^{s\tau} g(t-\tau)\,\mathrm{d}\tau \right) \mathrm{d}s.
\end{aligned}$$

Note now that the function

$$y(t;s) := \int_0^t e^{s\tau} g(t-\tau)\mathrm{d}\tau \tag{4.1}$$

is the unique solution to the initial value problem (with values in X)

$$\left[\begin{array}{l} y' = s\,y + g \quad t \geq 0, \\ \\ y(0) = 0. \end{array}\right. \tag{4.2}$$

The expression (4.1) is just the variation of constants formula for the initial value problem (4.2).

Let us now consider a constant time-step grid in $[0,\infty)$

$$0 = t_0 < t_1 < \ldots < t_n < \ldots, \qquad t_n := n\,\kappa.$$

We apply the backwards Euler method to (4.2)

$$\frac{y_n - y_{n-1}}{\kappa} = s\,y_n + g_n, \qquad g_n := g(t_n).$$

Therefore

$$y_n = y_n(s) = \frac{1}{1-\kappa\,s} y_{n-1} + \frac{\kappa}{1-\kappa\,s} g_n.$$

Instead of taking $y_0 = 0$, we will take $y_{-1} = 0$ as starting value (which fits with the idea of causality we have been exploring in previous chapters). Then, elementary algebra shows that

$$y_n(s) = \kappa \sum_{m=0}^{n} \frac{1}{(1-\kappa\,s)^{m+1}} g_{n-m},$$

which is a discrete version of

$$\int_0^{t_n} e^{s\tau} g(t_n - \tau)\mathrm{d}\tau.$$

Hence

$$\begin{aligned} (f * g)(t_n) &= \frac{1}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \mathrm{F}(s) y(t_n;s)\mathrm{d}s \\ &\approx \frac{1}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \mathrm{F}(s) y_n(s)\mathrm{d}s \\ &= \sum_{m=0}^{n} \left(\frac{\kappa}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \frac{1}{(1-\kappa\,s)^{m+1}} \mathrm{F}(s)\mathrm{d}s \right) g_{n-m}. \end{aligned}$$

We now turn our attention to what lies inside the bracket in the last expression

$$\begin{aligned} \omega_m(\kappa) &:= \frac{\kappa}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \frac{1}{(1-\kappa\,s)^{m+1}} \mathrm{F}(s)\mathrm{d}s \\ &= \frac{(-1)^m}{\kappa^m} \frac{-1}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \frac{1}{(s-\frac{1}{\kappa})^{m+1}} \mathrm{F}(s)\mathrm{d}s \\ &= \frac{(-1)^m}{\kappa^m} \frac{1}{m!} \mathrm{F}^{(m)}\left(\frac{1}{\kappa}\right) \\ &= \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}\zeta^m} \left(\mathrm{F}\left(\frac{1-\zeta}{\kappa} \right) \right) \Big|_{\zeta=0}. \end{aligned}$$

We have thus approximated the convolution $f * g$ at the points t_n by a discrete convolution

$$(f * g)(t_n) \approx \sum_{m=0}^{n} \omega_m(\kappa) g_{n-m}, \qquad g_n := g(t_n) \tag{4.3}$$

where

$$\mathrm{F}\left(\frac{1-\zeta}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m(\kappa)\zeta^m. \tag{4.4}$$

As written in (4.3)-(4.4) the only needs to apply the method (even if it does not give a good result!) to a particular convolution $f * g$ are the values of g at the discrete time values and the possibility of obtaining the Taylor expansion (4.4).

Generalization. Assume that $p(\zeta)$ is analytic in a neighborhood of the origin and that F is such that we can expand

$$\mathrm{F}\left(\frac{p(\zeta)}{k}\right) = \sum_{m=0}^{\infty} \omega_m(k)\zeta^m. \tag{4.5}$$

In this case, (4.3) defines a new method to approximate convolutions. From the point of view of approximation we will demand that $p(\zeta) \approx \log \zeta$ in a sense to be determined and that p is analytic near the origin. However, at this stage this is not needed. Since out kind of $\mathrm{F}(s)$ is defined on $\mathbb{C}_+$, and we need the method for $\kappa \to 0$ (κ is the time-step and we want it to be small), a reasonable hypothesis is

$$\mathrm{Re}\, p(\zeta) > 0, \qquad |\zeta| < c_0.$$

Composition of discrete convolutions. Assume that we have two different functions f_1 and f_2 and we have expanded

$$\mathrm{F}_1\left(\frac{p(\zeta)}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m^1(\kappa)\zeta^m, \qquad \mathrm{F}_2\left(\frac{p(\zeta)}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m^2(\kappa)\zeta^m.$$

It is then simple to see that for any sequence $\{g_n \,:\, n \geq 0\}$,

$$\sum_{m=0}^{n} \omega_m^1(\kappa)\left(\sum_{\ell=0}^{n-m} \omega_\ell^2(\kappa) g_{n-m-\ell}\right) = \sum_{m=0}^{n} \omega_m^{12}(\kappa) g_{n-m}, \qquad \forall n$$

where

$$\omega_m^{12}(\kappa) := \sum_{\ell=0}^{m} \omega_\ell^1(\kappa)\omega_{m-\ell}^2(\kappa)$$

is the convolution of the discrete sequences $\{\omega_m^1(\kappa)\}$ and $\{\omega_m^2(\kappa)\}$ and at the same time the sequence of weights associated to the symbol $s \mapsto \mathrm{F}_1(s)\mathrm{F}_2(s)$.

4.2 The ζ transform

Let X be Banach space and $\mathbf{h} := (h_n)_{n=0}^{\infty}$ be any sequence of elements of X. Consider the formal series

$$\mathrm{H}(\zeta) := \sum_{m=0}^{\infty} h_m \zeta^m.$$

Since it is a power series in the variable ζ, when it converges, it does so in an open ball in the complex plane centered at the origin and possibly in some points of its boundary. In that case, the elements of the sequence can be recovered by noticing that

$$h_m = \frac{1}{m!}\mathrm{H}^{(m)}(0),$$

since H is analytic in a neighborhood of the origin. In any case, we do not demand that this series converges. Constant series

$$\mathrm{H}(\zeta) \equiv h_0$$

correspond to almost trivial sequences $(h_0, 0, \ldots, 0, \ldots)$.

If the convolution of two sequences $\mathbf{h}^1$ and $\mathbf{h}^2$ is well-defined,

$$(\mathbf{h}^1 * \mathbf{h}^2)_n = \sum_{m=0}^{n} h^1_m h^2_{n-m},$$

then the ζ-series associated to this convolution is simply the Cauchy product of the formal series $\mathrm{H}_1(\zeta)$ and $\mathrm{H}_2(\zeta)$. The concern for the correct definition of the convolution product is not referred to convergence issues but to the fact that the products $h^1_m h^2_{m-n}$ make sense, which is the case when:

(a) $\mathbf{h}^1$ takes values in $\mathcal{B}(X,Y)$ and $\mathbf{h}^2$ in $\mathcal{B}(Z,X)$ or,

(b) $\mathbf{h}^1$ takes values in $\mathcal{B}(X,Y)$ and $\mathbf{h}^2$ takes values in X.

Let $(a_n) \subset \mathcal{B}(X,Y)$ and let a_0 be invertible. Consider the sequence defined by the recurrence

$$\begin{aligned} a_0^{\mathrm{inv}} &= a_0^{-1} \\ a_n^{\mathrm{inv}} &= -\left(\sum_{m=0}^{n-1} a_m^{\mathrm{inv}} a_{n-m}\right) a_0^{-1} \end{aligned}$$

and let

$$\mathrm{A}(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad \mathrm{A}^{-1}(\zeta) = \sum_{n=0}^{\infty} a_n^{\mathrm{inv}} \zeta^n.$$

Then

$$\mathrm{A}^{-1}(\zeta)\mathrm{A}(\zeta) = I_X, \qquad \mathrm{A}(\zeta)\mathrm{A}^{-1}(\zeta) = I_Y$$

where I_X and I_Y are the respective identity operators in X and Y and, at the same time, the corresponding constant series.

In case $\mathrm{A}(\zeta)$ is convergent –that is, when it defines an analytic function $B(0,r) \to \mathcal{B}(X,Y)$–, then if $\mathrm{A}(0) = a_0$ is invertible, $\mathrm{A}(\zeta)^{-1}$ exists for ζ sufficiently small and $\zeta \mapsto \mathrm{A}(\zeta)^{-1}$ is again analytic. The corresponding Taylor series is the same as the formal inversion of the series.

4.3 Convolution equations

Assume now that we are interested in solving a convolution equation

$$f * g = h,$$

where f and h are known: f is the operator part of the equation, h is the right-hand side. If

$$\mathrm{F}\left(\frac{p(\zeta)}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m(\kappa)\zeta^m$$

then we can try to solve the following triangular scheme of equations

$$\sum_{m=0}^{n} \omega_m(\kappa) g_{n-m} = h(t_n), \qquad n \geq 0.$$

Obviously this is equivalent to solving

$$\omega_0(\kappa) g_n = h(t_n) - \sum_{m=1}^{n} \omega_m(\kappa) g_{n-m}, \tag{4.6}$$

which is possible if $\omega_0(\kappa)$ is invertible. In the case

$$f : [0,\infty) \to \mathcal{B}(X,Y), \qquad h : [0,\infty) \to Y$$

the method provides a sequence of elements of X, $\{g_n\}$, that we expect to approximate the sequence $\{g(t_n)\}$ if this one exists. All the steps in (4.6) involve solving an operator equation with data in Y and solution in X, always with the same operator: $\omega_0(\kappa)$.

Proposition 4.3.1. *Assume that* $\mathrm{F}^{-1}(s)$ *exists for all* $s \in \mathbb{C}_+$ *and that*

$$\mathrm{F}\left(\frac{p(\zeta)}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m(\kappa)\zeta^m, \qquad \mathrm{F}^{-1}\left(\frac{p(\zeta)}{\kappa}\right) = \sum_{m=0}^{\infty} \omega_m^{\mathrm{inv}}(\kappa)\zeta^m.$$

Then the sequence defined by (4.6) *coincides with the sequence*

$$g_n := \sum_{m=0}^{n} \omega_m^{\mathrm{inv}}(\kappa) h(t_{n-m}).$$

Proof. Since $\mathrm{F}(\frac{1}{\kappa}p(0)) = \omega_0(\kappa)$ is invertible if $\operatorname{Re} p(0) > 0$, then (4.6) defines a sequence g_n. On the other hand

$$\mathrm{F}\left(\frac{p(\zeta)}{\kappa}\right)\mathrm{F}^{-1}\left(\frac{p(\zeta)}{\kappa}\right) = \mathrm{I}, \qquad |\zeta| \approx 0.$$

This implies that

$$\sum_{m=0}^{n} \omega_m(\kappa) \left[\sum_{\ell=0}^{n-m} \omega_\ell^{\mathrm{inv}}(\kappa) h(t_{n-m-\ell})\right] = h(t_n), \qquad \forall n.$$

By uniqueness of the equations (4.6), this sequence has to be $\{g_n\}$. $\square$

Assume that $p(\zeta)$ is a rational function

$$p(\zeta) = \frac{\alpha(\zeta)}{\beta(\zeta)} = \frac{\alpha_0 + \alpha_1\zeta + \ldots + \alpha_N\zeta^N}{\beta_0 + \beta_1\zeta + \ldots + \beta_N\zeta^N}, \tag{4.7}$$

with

$$\alpha_0\,\beta_0 \neq 0.$$

We consider an implicit $N-$step method to discretize

$$y' = f(t, y)$$

defined by the implicit equations

$$\begin{aligned}&\alpha_0 y_n + \alpha_1 y_{n-1} + \ldots + \alpha_N y_{n-N} \\ &\qquad = \kappa\Big(\beta_0 f(t_n, y_n) + \beta_1 f(t_{n-1}, y_{n-1}) + \ldots + \beta_N f(t_{n-N}, y_{n-N})\Big), \qquad n \geq 0.\end{aligned} \tag{4.8}$$

Then we can prove two results:

Proposition 4.3.2. *If we derive the CQ-method by approximating*

$$y(t_n; s) = \int_0^{t_n} e^{s\,\tau} g(t_n - \tau)\mathrm{d}\tau$$

with the multistep method (4.8) *for the equation*

$$\left[\begin{array}{l} y' = s\,y + g, \quad t \geq 0, \\ y(0) = 0, \end{array}\right.$$

and zero starting values

$$y_{-1} = \ldots = y_{-N} = 0,$$

then we obtain the CQ method associated to p.

Proof. Specialized to the equation $y' = s\,y + g$, the multistep method is

$$\begin{aligned}&\alpha_0 y_n + \alpha_1 y_{n-1} + \ldots + \alpha_N y_{n-N} \\ &\qquad = \kappa\, s\left[\beta_0 y_n + \beta_1 y_{n-1} + \ldots + \beta_N y_{n-N}\right] + \kappa\big(\beta_0 g_n + \beta_1 g_{n-1} + \ldots + \beta_N g_{n-N}\big).\end{aligned}$$

The sequence is well defined if and only if $\alpha_0 - \kappa\, s\,\beta_0 \neq 0$, that is,

$$\tfrac{1}{\kappa}p(0) \neq s.$$

Taking as starting values $y = -1 = \ldots = y_{-N} = 0$ and using $g_{-1} = \ldots = g_{-N} := 0$, this corresponds to

$$\alpha(\zeta)\mathrm{Y}(\zeta) = \kappa\, s\,\beta(\zeta)\mathrm{Y}(\zeta) + \kappa\mathrm{G}(\zeta).$$

The formal series $\alpha(\zeta) - \kappa\, s\,\beta(\zeta)$ is invertible and

$$\mathrm{Y}(\zeta) = \mathrm{Y}(\zeta; s) = \big(\alpha(\zeta) - \kappa s\beta(\zeta)\big)^{-1}\kappa\,\beta(\zeta)\mathrm{G}(\zeta) = \big(\tfrac{1}{\kappa}p(\zeta) - s\big)^{-1}\mathrm{G}(\zeta).$$

Notice that $p(\zeta)$ is holomorphic near the origin and that $\frac{1}{\kappa}p(\zeta) - s$ is holomorphic and non-vanishing near $\zeta = 0$.

For ζ small

$$\frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty}\left(\tfrac{1}{\kappa}p(\zeta)-s\right)^{-1}\mathrm{F}(s)\mathrm{d}s = \mathrm{F}\left(\tfrac{1}{\kappa}p(\zeta)\right)$$

and up to convergence questions, that is essentially all. □

Proposition 4.3.3. *If we apply the CQ method based on*

$$p(\zeta) = \frac{\alpha(\zeta)}{\beta(\zeta)} = \frac{\alpha_0 + \alpha_1\zeta + \ldots + \alpha_N\zeta^N}{\beta_0 + \beta_1\zeta + \ldots + \beta_N\zeta^N},$$

to an equation

$$(\delta_0 \otimes A) * g + (\delta_0' \otimes B) * g = h,$$

(where A and B are bounded operators) we obtain the same result as when applying the multisptep method to the implicit differential equation

$$Ag + \tfrac{\mathrm{d}}{\mathrm{d}t}(Bg) = h.$$

4.4 A more abstract point of view

The convolution quadrature is actually a convolution at the continuous level. Let start from the beginning. We need p defined in a neighborhood of the origin such that $\mathrm{Re}\, p(\zeta)$ for $|\zeta|$ small enough.

Assume that $\mathrm{F} : \mathbb{C}^+ \to \mathcal{B}(X,Y)$ is analytic. Then, the sequence of weights $\{\omega_m(\kappa)\}$ is well defined with the Taylor expansion

$$\mathrm{F}\Big(\frac{p(\zeta)}{\kappa}\Big) = \sum_{m=0}^{\infty}\omega_m(\kappa)\zeta^m. \tag{4.9}$$

We can then define the $\mathcal{B}(X,Y)$-valued causal (possibly non tempered) distribution:

$$f_\kappa := \sum_{m=0}^{\infty}\delta_{t_m}\otimes\omega_m(\kappa). \tag{4.10}$$

In this case, it is possible to apply the convolution of f_κ with any causal X-valued distribution g. This is how:

$$f_\kappa * g = \sum_{m=0}^{\infty}\omega_m(\kappa)g(\,\cdot\, - t_m).$$

If $g : \mathbb{R} \to X$ is a continuous casual function (as such, $g(0) = 0$), then

$$(f_\kappa * g)(t) = \sum_{m=0}^{\infty}\omega_m(\kappa)g(t-t_m) = \sum_{m=0}^{\lfloor t/\kappa\rfloor}\omega_m(\kappa)g(t-t_m)$$

is a causal Y-valued continuous function. We can then evaluate this convolution at the mesh points

$$(f_\kappa * g)(t_n) = \sum_{m=0}^{n} \omega_m(\kappa) g(t_{n-m}).$$

If we take the Laplace transform (assuming we can do it) in (4.10) we obtain

$$\begin{aligned} \mathrm{F}_\kappa(s) &= \sum_{m=0}^{\infty} e^{-st_m} \omega_m(\kappa) = \sum_{m=0}^{\infty} (e^{-s\kappa})^m \omega_m(\kappa) \\ &= \mathrm{F}\Big(\frac{p(e^{-s\kappa})}{\kappa}\Big). \qquad \text{(by (4.9))} \end{aligned}$$

We will see how, under some hypotheses on $p(\zeta)$, the approximation in the Laplace domain

$$\mathrm{F}(s) \longmapsto \mathrm{F}\big(\tfrac{p(e^{-s\kappa})}{\kappa}\big)$$

defines a transformation of symbols of the classes we studied in Chapter 3. This will justify that f_κ is a causal tempered distribution (and more) and that convolution quadrature can be viewed as discrete convolution (with sequences and ζ-transforms) or as continuous convolution (with distributions and Laplace transforms).

4.5 Symbol approximation by CQ

The building function. We need $p : U \to \mathbb{C}$ holomorphic in U, where $\{\zeta \in \mathbb{C} : |\zeta| \le 1\} \subset U$. Note that this precludes the existence of any kind of poles of p in the closed unit ball. Two main assumptions are made:

(a) $p : B(0,1) \to \mathbb{C}_+$.

(b) There exist $q \ge 1$, $C_0 > 0$ and $\kappa_0 > 0$ such that

$$\left| \frac{p(e^{-\kappa})}{\kappa} - 1 \right| \le C_0 \kappa^q \qquad \forall \kappa \le \kappa_0.$$

In particular $p(1) = 0$ and $p'(1) = -1$.

Unfortunately (because q is going to be the optimal order of the method), the previous hypotheses impose that $q \le 2$. (In the numerical ODE literature, this result is known as Dahlquist's Second Barrier.) Only three properties of this function will be used. They are presented in the following result. We will keep the notation of Chapters 2 and 3:

$$\sigma := \operatorname{Re} s \qquad \underline{\sigma} := \min\{1, \sigma\}.$$

The proof of this result uses standard techniques of complex analysis. It is included in Section 4.7.

Proposition 4.5.1. *If p satisfies the above properties and $\kappa_{\max} > 0$, there exist three positive constants C_1, C_2, C_3 such that for all $s \in \mathbb{C}_+$*

$$|p(e^{-s})| \le C_1|s| \qquad |p(e^{-s}) - s| \le C_2|s|^{q+1} \tag{4.11}$$

and

$$C_3\underline{\sigma} \le \mathrm{Re}\left(\frac{p(e^{-s\kappa})}{\kappa}\right) \qquad \forall \kappa \le \kappa_{\max}. \tag{4.12}$$

Only the constant C_3 depends on $\kappa_{\max}$.

The next result shows that the discrete convolution operator f_κ is actually a convolution operator. Given a symbol F, we can define the approximate symbol

$$\mathrm{F}_\kappa(s) := \mathrm{F}\left(\frac{p(e^{-s\kappa})}{\kappa}\right).$$

Analysis of this new symbol and its approximation properties to the original symbol $\mathrm{F}(s)$ is sketched next. Note that $\mathrm{F}_\kappa(s + \imath\kappa^{-1}) = \mathrm{F}_\kappa(s)$. This periodicity property reflects the fact that f_κ is discrete.

Proposition 4.5.2. *For $\mathrm{F} \in \mathcal{A}(\mu, X)$ with $\mu \ge 0$,*

$$\|\mathrm{F}_\kappa(s)\| \le C_1^\mu C_\mathrm{F}(C_3\underline{\sigma})|s|^\mu.$$

Therefore F_κ is a symbol of order μ with a bounding function independent of κ and $f_\kappa \in$ $\mathrm{TD}(X)$. Consequently

$$\|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| \le (1 + C_1^\mu)C_\mathrm{F}(\underline{C_3}\,\underline{\sigma})|\sigma|^\mu \qquad \forall s \in \mathbb{C}_+. \tag{4.13}$$

Proof. Applying the definition, it follows that

$$\begin{aligned}
\|\mathrm{F}_\kappa(s)\| &\le C_\mathrm{F}\Bigg(\underbrace{\mathrm{Re}\left(\frac{p(e^{-s\kappa})}{\kappa}\right)}_{\ge C_3\underline{\sigma}}\Bigg)\left|\frac{p(e^{-s\kappa})}{\kappa}\right|^\mu && \text{(by (4.12))}\\
&\le C_\mathrm{F}(C_3\underline{\sigma})\left|\frac{p(e^{-s\kappa})}{s\kappa}\right|^\mu |s|^\mu && C_\mathrm{F} \text{ is non-increasing}\\
&\le C_\mathrm{F}(C_3\underline{\sigma})C_1^\mu|s|^\mu. && \text{(by (4.11))}
\end{aligned}$$

The final bound follows from simple arguments:

$$\begin{aligned}
|s|^{-\mu}\|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| &\le C_\mathrm{F}(\sigma) + C_1^\mu C_\mathrm{F}(C_3\underline{\sigma})\\
&\le C_\mathrm{F}(\underline{\sigma}) + C_1^\mu C_\mathrm{F}(C_3\underline{\sigma})\\
&\le (1 + C_1^\mu)C_\mathrm{F}(\min\{1, C_3\}\,\underline{\sigma}).
\end{aligned}$$

□

Proposition 4.5.3. *If $\mathrm{F} \in \mathcal{A}(\mu, X)$, then $\mathrm{F}' \in \mathcal{A}(\mu, 0)$. In the case $\mu \ge 0$ we can bound*

$$\|\mathrm{F}'(s)\| \le \left(\tfrac{3}{2}\right)^\mu \tfrac{2}{\sigma} C_\mathrm{F}(\tfrac{\sigma}{2})|s|^\mu.$$

Proof. Let $\Xi(s) := \{z \in \mathbb{C} : |z-s| \le \sigma/2\}$ with positive orientation. We can then write

$$\mathrm{F}'(s) = \frac{1}{2\pi\imath}\int_{\Xi(s)} (z-s)^{-2}\mathrm{F}(z)\mathrm{d}z,$$

and bound using that $|\Xi(s)| = \pi\sigma$, $\mathrm{Re}\, z \ge \sigma/2$ and $|z| \le \frac{3}{2}|s|$ for all $z \in \Xi(s)$. □

Proposition 4.5.4. *Let* $\mathrm{F} \in \mathcal{A}(\mu, X)$ *with* $\mu \ge 0$. *Then*

$$\|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| \le A\, D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\,\kappa^q |s|^{\mu+q+1} \qquad \forall s \in \mathbb{C}_+$$

where

$$D_{\mathrm{F}}(\sigma) := C_{\mathrm{F}}(\tfrac{1}{2}\underline{C_3}\,\underline{\sigma})$$

The constant A *depends of* $p(\zeta)$ *(through the quantities of Proposition 4.5.1) and on* μ.

Proof. Recall that

$$\mathrm{F}(s) - \mathrm{F}_\kappa(s) = \mathrm{F}(s) - \mathrm{F}\left(\tfrac{p(e^{-s\kappa})}{\kappa}\right)$$

and let $\Xi_\kappa(s) := \{\alpha s + (1-\alpha)\frac{p(e^{-s\kappa})}{\kappa} : 0 \le \alpha \le 1\}$, be the straight line path than joins s and $\frac{p(e^{-s\kappa})}{\kappa}$. For $z \in \Xi_\kappa(s)$, we can bound

$$\mathrm{Re}\, z \ge \min\{\sigma, \mathrm{Re}\left(\frac{p(s^{-s\kappa})}{\kappa}\right)\} \ge \min\{\sigma, C_3\underline{\sigma}\} \ge \min\{1, C_3\}\underline{\sigma} = \underline{C_3\sigma} \tag{4.14}$$

and

$$|z| \le \max\{|s|, \left|\tfrac{p(e^{-s\kappa})}{\kappa}\right|\} \le |s|\max\{1, C_1\}. \tag{4.15}$$

Using (4.14) and (4.15) in Proposition 4.5.3, it follows that

$$\begin{aligned}\|\mathrm{F}'(z)\| &\le (\tfrac{3}{2})^\mu \frac{2}{\mathrm{Re}\, z} C_{\mathrm{F}}\left(\tfrac{1}{2}\mathrm{Re}\, z\right)|z|^\mu \\ &\le \tfrac{2}{\underline{C}_3}\left(\tfrac{3}{2}\max\{1, C_1\}\right)^\mu \tfrac{1}{\underline{\sigma}} C_{\mathrm{F}}\left(\tfrac{1}{2}\underline{C_3\sigma}\right)|s|^\mu \qquad \forall z \in \Xi_\kappa(s)\end{aligned}$$

this, the fact that

$$\left|s - \frac{p(e^{-s\kappa})}{\kappa}\right| = \kappa^{-1}\left|s\kappa - p(e^{-s\kappa})\right| \le C_2\kappa^q|s|^{q+1},$$

and

$$\|\mathrm{F}(s) - \mathrm{F}\left(\tfrac{p(e^{-s\kappa})}{\kappa}\right)\| \le \sqrt{2}\left|s - \frac{p(e^{-s\kappa})}{\kappa}\right| \max_{z \in \Xi_\kappa(s)} \|\mathrm{F}'(z)\|$$

(see Exercises) prove the result. □

Proposition 4.5.5 (L^1 estimates)**.** *Let* $r > 0$ *and* $\mathrm{F} \in \mathcal{A}(\mu, X)$ *with* $\mu \ge 0$. *Then*

$$\|(\,\cdot\,)^{-1-\mu-r}(\mathrm{F} - \mathrm{F}_\kappa)\|_{L^1(\sigma+\imath\mathbb{R},X)} \le C\, D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\max\{\underline{\sigma}^{-r}, \sigma^{q-r}\}\, E(\kappa),$$

where

$$E(\kappa) := \kappa^{r\frac{q}{q+1}} + \delta_{r,q+1}\kappa^q|\log\kappa| + \kappa^q$$

and the constant C *depends on* $p(\zeta)$, μ *and* r.

Proof. We can gather the approximation properties proved in Propositions 4.5.2 and 4.5.4 in the inequality

$$\|s^{-1-\mu-r}(\mathrm{F}(s)-\mathrm{F}_\kappa(s))\| \le B\, D_{\mathrm{F}}(\sigma) \begin{cases} |s|^{-r-1}, \\ \underline{\sigma}^{-1}|s|^{q-r}\kappa^q, \end{cases} \tag{4.16}$$

where the constant B depends on μ and on $p(\zeta)$. If $s=\sigma+\imath\eta$ we can always bound

$$\tfrac{1}{\sqrt{2}}\underline{\sigma}(1+|\eta|) \le |s| \le \max\{1,\sigma\}(1+|\eta|).$$

Therefore

$$\begin{aligned} |s|^{-r-1} &\le 2^{\frac{r+1}{2}}\,\underline{\sigma}^{-r-1}(1+|\eta|)^{-r-1}, & \\ |s|^{q-r} &\le 2^{\frac{r-q}{2}}\underline{\sigma}^{q-r}(1+|\eta|)^{q-r}, & q-r\le 0, \\ |s|^{q-r} &\le \max\{1,\sigma\}^{q-r}(1+|\eta|)^{q-r} & q-r>0. \end{aligned}$$

We can collect the last two inequalities in the following condensed form and :

$$\begin{aligned} |s|^{q-r} &\le 2^{\frac{r-q}{2}}\max\{\underline{\sigma}^{-r},\sigma^{q-r}\}\,(1+|\eta|)^{q-r}, \\ |s|^{-r-1} &\le 2^{\frac{r+1}{2}}\,\underline{\sigma}^{-1}\max\{\underline{\sigma}^{-r},\sigma^{q-r}\}\,(1+|\eta|)^{-r-1}. \end{aligned}$$

Bringing these inequalities to (4.16) and collecting constants again, we can write

$$\begin{aligned} \|s^{-1-\mu-r}(\mathrm{F}(s)-\mathrm{F}_\kappa(s))\| &\le C\,D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\max\{\underline{\sigma}^{-r},\sigma^{q-r}\} \\ &\qquad \times\min\{(1+|\eta|)^{-r-1},\kappa^q(1+|\eta|)^{q-r}\}, \end{aligned}$$

with C depending on B (and thus on p and μ) and on the index r. Since for $r,q>0$

$$\int_{-\infty}^{\infty}\min\{(1+|\eta|)^{-r-1},\kappa^q(1+|\eta|)^{q-r}\}\mathrm{d}\eta \le C\times E(\kappa),$$

the result follows readily. □

Proposition 4.5.6 (Pointwise estimates)**.** *Let* $\mathrm{F}\in\mathcal{A}(\mu,X)$ *with* $\mu\ge 0$.

(a) *If* r,ε *satisfy*

$$0\le\varepsilon\le r-(q+1),$$

then

$$\|\mathrm{F}(s)-\mathrm{F}_\kappa(s)\| \le C|s|^{\mu+r-\varepsilon}D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\sigma^{q+1-r+\varepsilon}\,\kappa^q \qquad \forall s\in\mathbb{C}_+.$$

(b) *If* r,ε *satisfy*

$$0\le\varepsilon\le r\le q+1,$$

then

$$\|\mathrm{F}(s)-\mathrm{F}_\kappa(s)\| \le C|s|^{\mu+r-\varepsilon}D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\,\kappa^{(r-\varepsilon)\frac{q}{q+1}} \qquad \forall s\in\mathbb{C}_+.$$

The constants C *depend of* μ,p *and* r.

Proof. Using Propositions 4.5.2 and 4.5.4 (see also (4.16)) we can easily bound

$$\begin{aligned} \|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| &\le B\,|s|^\mu D_{\mathrm{F}}(\sigma) \min\{1, |s|^{q+1}\kappa^q \underline{\sigma}^{-1}\} \\ &= \underbrace{B|s|^\mu D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}}_{=\Xi} \min\{1, \sigma, |s|^{q+1}\kappa^q\} \end{aligned}$$

With the hypotheses (a) for the parameters, we continue the bounds as

$$\|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| \le \Xi|s|^{q+1}\kappa^q \le \Xi|s|^{r-\varepsilon}\sigma^{q+1-(r-\varepsilon)}\kappa^q,$$

due to the fact that $q + 1 - r + \varepsilon \le 0$.

In the case of (b), we have that $(r - \varepsilon)/(q + 1) \le 1$, and therefore

$$\min\{1, \alpha\} \le \alpha^{\frac{r-\varepsilon}{q+1}} \qquad \forall \alpha > 0,$$

We proceed as before to bound

$$\|\mathrm{F}(s) - \mathrm{F}_\kappa(s)\| \le \Xi \min\{1, |s|^{q+1}\kappa^q\} \le \Xi|s|^{r-\varepsilon}\kappa^{(r-\varepsilon)\frac{q}{q+1}},$$

which finishes the proof. □

4.6 Convergence of CQ

Proposition 4.6.1 (Uniform convergence with L^1 regularity)**.** *Let* $\mathrm{F} \in \mathcal{A}(\mu, \mathcal{B}(X, Y))$ *with* $\mu \ge 0$ *and define* $D_{\mathrm{F}}(\sigma) := C_{\mathrm{F}}(\frac{1}{2}\underline{C}_3\,\underline{\sigma})$. *Let g be a causal X-valued function such that*

$$g \in \mathcal{C}^{k-1}(\mathbb{R}, X) \qquad g^{(k)} \in L^1(\mathbb{R}, X), \qquad k > \mu + 1.$$

Then

$$\|(f * g)(t) - (f_\kappa * g)(t)\| \le D \times \mathrm{error}(\kappa)\, h(t) \int_0^t \|g^{(k)}(\tau)\|\mathrm{d}\tau,$$

where D depends only on μ, k and p,

$$\mathrm{error}(\kappa) := \kappa^{(k-\mu-1)\frac{q}{q+1}} + \delta_{k,\mu+q+2}\kappa^q|\log\kappa| + \kappa^q$$

and

$$h(t) := \begin{cases} t^{k-\mu} D_{\mathrm{F}}(t^{-1}) & t \ge 1, \\ D_{\mathrm{F}}(1)t^{k-\mu-1-q} & t \le 1 \quad \text{and} \quad k \ge \mu + 1 + q, \\ D_{\mathrm{F}}(1), & t \le 1 \quad \text{and} \quad \mu + 1 < k < \mu + 1 + q. \end{cases}$$

Optimal convergence (order q) is obtained for $k > \mu + q + 2$.

Proof. The proof uses techniques that we have already met in the analysis of convolution operators. (See Section 3.2 and especially Proposition 3.2.2.) The Laplace transform of the error $f * g - f_\kappa * g$ is

$$(\mathrm{F} - \mathrm{F}_\kappa)\,\mathrm{G} \in \mathcal{A}(\mu - k, Y) \qquad \mu - k < -1.$$

(To see this, use Proposition 4.5.2 for the term $\mathrm{F}-\mathrm{F}_\kappa$ and recall the proof that $\mathrm{G}\in\mathcal{A}(-k,X)$ given in Proposition 3.2.2.) Since $\mu-k<-1$ we can apply the strong inversion formula for the Laplace transform (see Section 3.1 and in particular (3.2)), so that

$$\begin{aligned}\|(f*g)(t)-(f_\kappa*g)(t)\| &\le \frac{e^{\sigma t}}{2\pi}\|(\mathrm{F}-\mathrm{F}_\kappa)\mathrm{G}\|_{L^1(\sigma+\imath\mathbb{R},Y)}\\ &\le \frac{e^{\sigma t}}{2\pi}\|(\,\cdot\,)^{-k}(\mathrm{F}-\mathrm{F}_\kappa)\|_{L^1(\sigma+\imath\mathbb{R},\mathcal{L}(X,Y))}\sup_{\mathrm{Re}\,s=\sigma}\|s^k\mathrm{G}(s)\|.\end{aligned}$$

Using the integral formula for the Laplace transform of $s^k\mathrm{G}(s)$, it follows that

$$\sup_{\mathrm{Re}\,s=\sigma}\|s^k\mathrm{G}(s)\|\le\int_0^\infty e^{-\sigma\tau}\|g^{(k)}(\tau)\|\mathrm{d}\tau\le\int_0^\infty\|g^{(k)}(\tau)\|\mathrm{d}\tau.$$

Using Proposition 4.5.5 with $r=k-1-\mu$ and noticing that $\mathrm{error}(\kappa)=E(\kappa)$ for those parameters, we arrive at an error bound, proportional to

$$e^{\sigma t}D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\max\{\underline{\sigma}^{-k+1+\mu},\sigma^{q-k+1+\mu}\}\mathrm{error}(\kappa)\int_0^\infty\|g^{(k)}(\tau)\|\mathrm{d}\tau.$$

The dependence on t of the error constants is obtained by taking $\sigma=t^{-1}$ or $\sigma=1$ (this last value when $t\ge 1$ and $k-\mu-1-q<0$).

To change the integration interval for $\|g^{(k)}\|$ from $(0,\infty)$ to $(0,t)$ we can use the causality argument of Proposition 3.2.2. (See Exercises.) □

Remark. Note that the error term in Proposition 4.6.1 can be split into three different cases:

(a) If $\mu+1<k<\mu+q+2$, then

$$\mathrm{error}(\kappa)=\kappa^{\alpha\frac{q}{q+1}}\qquad\alpha:=k-\mu-1\in(0,q+1).$$

(b) If $k=\mu+q+2$ (which can only happen if μ is an integer), then $\mathrm{error}(\kappa)=\kappa^q|\log\kappa|$.

(c) Finally, for $k>\mu+q+2$, the error is of optimal order κ^q.

Proposition 4.6.2 (Uniform estimates with L^2 regularity)**.** *Let $\mathrm{F}\in\mathcal{A}(\mu,\mathcal{B}(X,Y))$ with $\mu\ge 0$ and define $D_{\mathrm{F}}(\sigma):=C_{\mathrm{F}}(\frac{1}{2}\underline{C}_3\,\underline{\sigma})$. Let g be a causal X-valued function such that*

$$g\in\mathcal{C}^{k-1}(\mathbb{R},X)\qquad g^{(k)}\in L^2(\mathbb{R},X).$$

(a) *If $k>\mu+q+\frac{3}{2}$, then*

$$\|(f*g)(t)-(f_\kappa*g)(t)\|\le\kappa^q\,h(t)\left(\int_0^t\|g^{(k)}(\tau)\|^2\mathrm{d}\tau\right)^{1/2},$$

where

$$h_1(t)=C\frac{2^{\alpha/4}}{\sqrt{\alpha}}D_{\mathrm{F}}(t^{-1})\max\{1,t\}t^\alpha\qquad\alpha=k-\mu-q-\tfrac{3}{2}.$$

(b) *If $\mu+q+\frac{3}{2}\ge k>\mu+\frac{1}{2}$, then*

$$\|(f*g)(t)-(f_\kappa*g)(t)\|\le \kappa^{\beta\frac{q}{q+1}}|\log\kappa|^{1/2}\,h(t)\Big(\int_0^t\|g^{(k)}(\tau)\|^2\mathrm{d}\tau\Big)^{1/2},$$

where

$$h_2(t)=C\frac{2^{\beta/4}}{\sqrt{\beta}}D_{\mathrm{F}}(t^{-1})\max\{1,t^{\beta+1}\}\qquad \beta=k-\mu-\tfrac{1}{2}\in(0,q+1].$$

The constants C depend on k,μ and p.

Proof. Let $e_\kappa:=f*g-f_\kappa*g$ and assume that g is smooth causal and compactly supported. Using the inversion formula for the Laplace transform (using the inversion contour $\mathrm{Re}\,s=\sigma$), we can bound

$$\begin{aligned}\|e_\kappa(t)\| &\le \frac{e^{\sigma t}}{2\pi}\Big(\int_{-\infty}^\infty\frac{\mathrm{d}\eta}{|\sigma+\imath\eta|^{\frac12+\varepsilon}}\Big)^{1/2}\sup_{\mathrm{Re}\,s=\sigma}|s|^{-k+\frac12+\varepsilon}\|\mathrm{F}(s)-\mathrm{F}_\kappa(s)\|\\ &\quad\times\Big(\int_{-\infty}^\infty\|(\sigma+\imath\eta)^k\mathrm{G}(\sigma+\imath\eta)\|^2\mathrm{d}\eta\Big)^{1/2}\\ &\le \frac{e^{\sigma t}}{2\pi}\Big(\frac{2^{\varepsilon+\frac12}}{\varepsilon\sigma^{2\varepsilon}}\Big)^{1/2}\sup_{\mathrm{Re}\,s=\sigma}|s|^{-k+\frac12+\varepsilon}\|\mathrm{F}(s)-\mathrm{F}_\kappa(s)\|\Big(\frac{1}{2\pi}\int_0^\infty\|g^{(k)}(\tau)\|^2\mathrm{d}\tau\Big)^{1/2},\end{aligned}$$

where in the last inequality we have used Plancherel's identity (see Section 3.7) and, for obvious reasons, we need to take $\varepsilon>0$.

In case (a), we apply Proposition 4.5.6(a) with

$$r=k-\mu-\tfrac12>q+1,\qquad \varepsilon=\tfrac12(r-(q+1))=\tfrac12\alpha>0$$

so that

$$\mu+r-\varepsilon=k-\tfrac12-\varepsilon\qquad\text{and}\qquad q+1-r+\varepsilon=-\tfrac12\alpha.$$

Then

$$\frac{2^{\frac{\varepsilon}{2}+\frac14}}{\sigma^\varepsilon\sqrt{\varepsilon}}\sup_{\mathrm{Re}\,s=\sigma}|s|^{-k+\frac12+\varepsilon}\|\mathrm{F}(s)-\mathrm{F}_\kappa(s)\|\le\frac{2^{3/4}}{\sqrt{\alpha}}\frac{2^{\frac12(\alpha+1)}}{\sigma^\alpha}D_{\mathrm{F}}(\sigma)\underline{\sigma}^{-1}\kappa^q.$$

The result follows now from taking $\sigma=1/t$, simplifying, using a density argument and causality to generalize for any g and restrict the integral in the right hand side to $(0,t)$.

Part (b) is left to the reader as an exercise. □

4.7 Appendix: A-acceptable functions

Let us recall the hypotheses on p. We assume that there is a open set U such that $\{\zeta\in\mathbb{C}\,:\,|\zeta|\le1\}\subset U$ and that $p:U\to\mathbb{C}$ is holomorphic and satisfies

(a) $p:B(0,1)\to\mathbb{C}_+$.

(b) There exist $q \geq 1$, $C_0 > 0$ and $\kappa_0 > 0$ such that

$$\left| \frac{p(e^{-\kappa})}{\kappa} - 1 \right| \leq C_0 \kappa^q \qquad \forall \kappa \leq \kappa_0.$$

In particular $p(1) = 0$ and $p'(1) = -1$.

Note that the functions $s \mapsto p(e^{-s})$ and $s \mapsto s_\kappa = \kappa^{-1} p(e^{-\kappa s})$ are holomorphic $\mathbb{C}_+ \to \mathbb{C}_+$.

Proposition 4.7.1. *There exists $C_1 > 0$ such that*

$$|p(e^{-s})| \leq C_1 |s| \qquad \forall s \in \mathbb{C}_+.$$

Proof. It is not complicated to prove that

$$\left| \frac{e^{-s} - 1}{s} \right| \leq 2 \qquad \forall s \in \mathbb{C}_+. \tag{4.17}$$

Since the function $p(\zeta)(1-\zeta)^{-1}$ is holomorphic in U, we can bound

$$\left| \frac{p(e^{-s})}{s} \right| \leq 2 \left| \frac{p(e^{-s})}{e^{-s} - 1} \right| \leq 2 \max_{|\zeta| \leq 1} \left| \frac{p(\zeta)}{\zeta - 1} \right|,$$

which completes the proof. □

Proposition 4.7.2. *There exists $C_2 > 0$ such that*

$$|p(e^{-s}) - s| \leq C_2 |s|^{q+1} \qquad \forall s \in \mathbb{C}_+.$$

Proof. The hypotheses of p imply that the function

$$h(s) := \frac{p(e^{-s}) - s}{s^{q+1}}$$

is holomorphic in a neighborhood of the origin and therefore, there exists $\rho > 0$ and $\widetilde{C} > 0$ such that such that

$$\left| \frac{p(e^{-s}) - s}{s^{q+1}} \right| \leq \widetilde{C}, \quad \textit{when} \quad |s| \leq \rho.$$

For $s \in \mathbb{C}_+$ such that $|s| \geq \rho$, we can bound using Proposition 4.7.1

$$|p(e^{-s}) - s| \leq (C_1 + 1)|s| \leq (C_1 + 1)\rho^{-q} |s|^{q+1}.$$

This finishes the proof. □

Proposition 4.7.3. *There exists $C_3 > 0$ such that*

$$\mathrm{Re}\, p(e^{-s}) \geq C_3 \underline{\sigma} \qquad \forall s \in \mathbb{C}_+.$$

Proof. By the Cauchy Representation Formula

$$p(z) = \frac{1}{2\pi\imath}\int_{|\zeta|=1}\frac{p(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi}\frac{p(e^{\imath\theta})}{e^{\imath\theta}-z}\,e^{\imath\theta}\mathrm{d}\theta \qquad \forall z \in B(0;1)$$

and therefore

$$\mathrm{Re}\,p(re^{\imath\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,p(e^{\imath\theta})\,\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta \qquad \forall r<1, \quad \forall \phi.$$

(This is just the well-known Poisson formula.) We now use the fact that $\mathrm{Re}p(e^{\imath\theta}) \geq 0$ (this follows from the hypotheses on p) and

$$\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2} \geq \frac{1-r^2}{1+2r+r^2} = \frac{1-r}{1+r}$$

to bound

$$\mathrm{Re}\,p(r\,e^{\imath\phi}) \geq \frac{1-r}{1+r}\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,p(e^{\imath\theta})\mathrm{d}\theta}_{=:C_p>0}.$$

Applying this to the point $\zeta = e^{-s} = e^{-\sigma}e^{-\imath\omega}$ for $s \in \mathbb{C}_+$, it follows that

$$\mathrm{Re}\,p(e^{-s}) \geq C_p\frac{1-e^{-\sigma}}{1+e^{-\sigma}}.$$

Finally

$$\frac{1-e^{-\sigma}}{1+e^{-\sigma}} = \tanh(\sigma/2) \geq \tanh(1/2)\min\{1,\sigma\} = \tanh(1/2)\underline{\sigma},$$

which completes the proof. □

Proposition 4.7.4. *For all $\kappa \leq \kappa_{\max}$*

$$\mathrm{Re}\left(\frac{p(e^{-\kappa\,s})}{\kappa}\right) \geq \frac{C_3}{\max\{1,\kappa_{\max}\}}\underline{\sigma} \qquad \forall s \in \mathbb{C}_+.$$

Proof. We just apply Proposition 4.7.4 to bound

$$\begin{aligned}\mathrm{Re}\,p(e^{-\kappa\,s}) &\geq C_3\min\{1,\kappa\sigma\} = C_3\kappa\min\{\kappa^{-1},\sigma\}\\ &\geq C_3\kappa\min\{\kappa^{-1},1\}\min\{1,\sigma\} \geq C_3\kappa\min\{\kappa_{\max}^{-1},1\}\underline{\sigma}.\end{aligned}$$

This finishes the proof. □

Proposition 4.7.5. *In the hypotheses above, any zero of p in the boundary of the unit disk is simple.*

Proof. See [10, Chapter 5, Theorem 1.5]. The result is obviously related to the fact that the unit ball is mapped to $\mathbb{C}_+$. A double zero would then make part of the unit disk to be mapped outside $\mathbb{C}_+$. □

4.8 Exercises

1. (Section 4.3) Prove Proposition 4.3.3.

2. (Section 4.5) The CQ method can also be described by means of a discrete differentiation operator
$$s_\kappa := \kappa^{-1} p(e^{-s\kappa})$$
that is used to define discrete versions of all symbols $\mathrm{F}_\kappa(s) := \mathrm{F}(s_\kappa)$. Show that Proposition 4.5.1 can be written in the alternative form
$$|s_\kappa| \le C_1 |s|, \qquad |s_\kappa - s| \le C_2 \kappa^q |s|^{q+1}, \qquad \operatorname{Re} s_\kappa \ge C_3 \underline{\sigma} \qquad \forall s \in \mathbb{C}_+.$$

3. (Section 4.5) Consider the discrete differentiation operator $\partial_\kappa u := \mathcal{L}^{-1}\{s_\kappa \mathrm{U}(s)\}$, where $s_\kappa := \kappa^{-1} p(e^{-s\kappa})$. Use the previous exercise and Proposition 3.2.2 to provide direct bounds for
$$\|(\partial_\kappa u)(t)\| \quad \text{and} \quad \|u'(t) - (\partial_\kappa u)(t)\|.$$

4. (Section 4.5) Prove that if F is holomorphic in a convex set, then for all s_1, s_2 in that set
$$\|\mathrm{F}(s_1) - \mathrm{F}(s_2)\| \le \sqrt{2} |s_1 - s_2| \max_{z \in [s_1, s_2]} \|\mathrm{F}'(z)\|,$$
where $[s_1, s_2] = \{\alpha s_1 + (1-\alpha) s_2 \,:\, \alpha \in [0,1]\}$. (**Hint.** Use the Mean Value Theorem for bivariate functions and the Cauchy-Riemann equations.)

5. (Section 4.6) Show that if f is a causal distribution and for sufficiently smooth causal g
$$\|(f * g)(t)\| \le C(t) \int_0^\infty \|g^{(k)}(\tau)\| \mathrm{d}\tau,$$
then
$$\|(f * g)(t)\| \le C(t) \int_0^t \|g^{(k)}(\tau)\| \mathrm{d}\tau.$$
(**Hint.** You can use the strategy of the proof of Proposition 3.2.2. The idea can be simplified to showing the result for $k = 0$ and then extending it for general k using differentiation and antidifferentiation operators.)

6. (Section 4.6) Prove part (b) of Proposition 4.6.1. (**Hint.** Use the same inequalities that are used to prove part (a). Assume that $\log \kappa < -1$ and take
$$\varepsilon := -\frac{r}{\log \kappa} < r, \qquad r = k - \mu - \tfrac{1}{2} = \beta$$
in Proposition 4.5.6. Note also that $t^{-r/\log \kappa} \le \max\{1, t^r\}$.)

7. (Section 4.7) Show that in the hypotheses of Section 4.7, the function
$$h(s) := \frac{p(e^{-s}) - s}{s^{q+1}}$$
is holomorphic in a neighborhood of zero.

Chapter 5

Convolution Quadrature and the single layer potential

5.1 Back to scattering problems

Let us then go back to the problem of scattering by a sound soft obstacle, using a single layer potential. Data is a causal function g with values in $H^{1/2}(\Gamma)$. An integral equation is solved to find a density

$$\mathcal{V} * \lambda = g \qquad \Longleftrightarrow \qquad \lambda = \mathcal{V}^{-1} * g$$

and then a potential is computed using λ:

$$u = \mathcal{S} * \lambda = \mathcal{S} * \mathcal{V}^{-1} * g.$$

Recall that (Proposition 2.6.1)

$$\|\mathrm{V}^{-1}(s)\| \le C\frac{|s|^2}{\sigma\underline{\sigma}}$$

and (Section 3.3)

$$\|\mathrm{S}(s)\mathrm{V}^{-1}(s)\|_{H^{1/2}(\Gamma)\to H^1(\mathbb{R}^d\setminus\Gamma)} \le C\frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}}.$$

Because we are first solving a convolution equation, we do not have access to the entire function $\lambda_\kappa := (\mathcal{V}^{-1})_\kappa * g$ but only to its values at the fixed time steps $t_n = \kappa\, n$. The error bound (Proposition 4.6.1) is applied to V^{-1} and to $\mathrm{S}\,\mathrm{V}^{-1}$ separately. For the BDF2-based method ($q = 2$) the errors

$$\|\lambda(t_n) - \lambda_\kappa(t_n)\|_{-1/2,\Gamma}$$

are bounded by a product of three quantities:

$$\mathrm{error}_\lambda(\kappa) \times \mathrm{behavior}_\lambda(t_n) \times \int_0^{t_n} \|g^{(k)}(\tau)\|\mathrm{d}\tau.$$

Let us tabulate the results for the possible values of k:

k	$\text{error}_\lambda(\kappa)$	$\text{behavior}_\lambda(t)$ for $t \ge 1$	$\text{behavior}_\lambda(t)$ for $t \le 1$
4	$\kappa^{2/3}$	t^4	1
5	$\kappa^{4/3}$	t^5	1
6	$\kappa^2 \lvert \log \kappa \rvert$	t^6	t
7	κ^2	t^7	t^2

The errors

$$\|u(t_n) - u_\kappa(t_n)\|_{1,\Omega^+}$$

are bounded again by a triple product

$$\text{error}_u(\kappa) \times \text{behavior}_u(t_n) \times \int_0^t \|g^{(k)}(\tau)\| \mathrm{d}\tau,$$

with the following table:

k	$\text{error}_u(\kappa)$	$\text{behavior}_u(t)$ for $t \ge 1$	$\text{behavior}_\lambda(t)$ for $t \le 1$
3	$\kappa^{1/3}$	t^4	1
4	κ	t^5	1
5	$\kappa^{5/3}$	t^6	$t^{1/2}$
6	κ^2	t^7	$t^{3/2}$

5.2 Full discretization with CQ-BEM

The method. A fully discrete method for the problem

$$\mathcal{V} * \lambda = g \qquad u = \mathcal{S} * \lambda$$

can be devised by first discretizing in space with a Galerkin scheme and then using convolution quadrature for the resulting system. In order to do this, we start by choosing a finite dimensional space $X_h \subset H^{-1/2}(\Gamma)$ (in practice, the space is always taken as $X_h \subset L^\infty(\Gamma)$). In the time domain, we look for $\lambda^h \in \mathrm{TD}(X_h)$ (where X_h is endowed with the $H^{-1/2}(\Gamma)$-norm) such that

$$\langle \mu^h, \mathcal{V} * \lambda^h \rangle_\Gamma = \langle \mu^h, g \rangle_\Gamma \qquad \forall \mu^h \in X_h. \tag{5.1}$$

It is not difficult to see that this is again a convolution equation: to do that, we take the (steady state) operator $R^h : H^{1/2}(\Gamma) \to X_h'$

$$H^{1/2}(\Gamma) \ni g \longmapsto R^h g := \langle \,\cdot\,, g \rangle_\Gamma : X_h \to \mathbb{R}$$

and note that the equation (5.1) is equivalent to

$$(\delta_0 \otimes R^h) * \mathcal{V} * \lambda^h = R^h g.$$

The operator to which convolution quadrature is applied is an operator $X_h \to X_h'$

$$\lambda^h \mapsto (\delta_0 \otimes R^h) * \mathcal{V} * \lambda^h = R^h(\mathcal{V} * \lambda^h).$$

In a postprocessing step, CQ is used again for the discretization of the potential

$$u^h := \mathcal{S} * \lambda^h. \tag{5.2}$$

A look at the equations. If $\{\Phi_1, \ldots, \Phi_N\}$ is a basis of X_h, then we can write

$$\lambda^h = \sum_{j=1}^{N} \lambda_j \otimes \Phi_j, \qquad \lambda_j \in \mathrm{TD}(\mathbb{R})$$

and, assuming the time distributions λ_j are functions, write the semidiscrete system (4.17) in the following practical form:

$$\sum_{j=1}^{N} \int_\Gamma \int_\Gamma \frac{\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \lambda_j(t - c^{-1}|\mathbf{x}-\mathbf{y}|)\, \mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma \Phi_i(\mathbf{x})\, g(\mathbf{x}, t)\mathrm{d}\Gamma(\mathbf{x})$$
$$i = 1, \ldots, N, \qquad t > 0.$$

The Laplace domain form of the operator in this system corresponds to the s-dependent matrix

$$\mathrm{V}_{ij}(s) := \int_\Gamma \int_\Gamma \frac{e^{-s|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\, \Phi_i(\mathbf{x})\Phi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}),$$

which is an X^h-based Galerkin discretization of the integral operator $\mathrm{V}(s)$. The convolution quadrature method then needs an expansion

$$\mathrm{V}_{ij}\big(\tfrac{p(\zeta)}{\kappa}\big) = \sum_{m=0}^{\infty} \mathrm{V}_{ij}^m(\kappa)\zeta^m.$$

Time-stepping is carried out by solving systems

$$\sum_{m=0}^{n} \sum_{j=1}^{N} \mathrm{V}_{ij}^m \lambda_j^{n-m} = g_i^n \qquad i = 1, \ldots, N, \qquad n \geq 0,$$

where

$$g_i^n := \int_\Gamma \Phi_i(\mathbf{x}) g(\mathbf{x}, t_n)\mathrm{d}\Gamma(\mathbf{x}) \qquad i = 1, \ldots, N, \qquad n \geq 0.$$

The unknowns allow us to reconstruct functions

$$X_h \ni \lambda_\kappa^h(t_n) := \sum_{j=1}^{N} \lambda_j^n \Phi_j \approx \lambda^h(t_n),$$

for which only the values at discrete times are computed. Every time step requires the solution of a $N \times N$ system

$$\sum_{j=1}^{N} \mathrm{V}_{ij}^0(\kappa)\lambda_j^n = g_i^n - \sum_{m=1}^{n} \sum_{j=1}^{N} \mathrm{V}_{ij}^m(\kappa)\lambda_j^{n-m}, \qquad i = 1, \ldots, N. \tag{5.3}$$

The matrix is

$$\mathrm{V}^0_{ij}(\kappa) = \int_\Gamma \int_\Gamma \frac{e^{-\frac{p(0)}{\kappa}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\Phi_i(\mathbf{x})\Phi_j(\mathbf{y})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{y}),$$

which is, once again, the result of using an X^h-based Galerkin discretization to the operator $\mathrm{V}(\frac{p(0)}{\kappa})$. Actually, if we apply directly a convolution quadrature method to $\mathcal{V} * \lambda = g$, we end up with a sequence of integral equations

$$\mathrm{V}^0(\kappa)\lambda^n = g(t^n) - \sum_{m=1}^{n} \mathrm{V}^m(\kappa)\lambda^{n-m}, \tag{5.4}$$

where

$$\mathrm{V}\big(\tfrac{p(\zeta)}{\kappa}\big) = \sum_{m=0}^{\infty} \mathrm{V}^m(\kappa)\zeta^m$$

is the sequence of weights associated to the CQ scheme. Using an X^h-based Galerkin discretization to each of the integral equations (5.4) is equivalent to the method we had just derived, that is, for this kind of problems, Galerkin semidiscretization is space and convolution quadrature in time commute.

Final words. The sequence of problems (5.3) (and the time-semidiscrete sequence (5.4)) present the serious disadvantage of having an infinite tail. In other words, the passage through the Laplace domain introduces a regularization of the wave equation that eliminates the Huygens' principle that so clearly appears in the time domain retarded operators and potentials. Some estimates exist concerning the size of the operators $\mathrm{V}^m(\kappa)$ for large enough m and on the effect of doing away with them in the time-stepping process. They provide theoretical support for a trick that can be safely played for in practice. Practical evaluation of the discrete kernels V^n_{ij} and the discrete convolutions where they appear can be accomplished with a battery of techniques based on the FFT and recursion. Details on how this is done can be found in [6].

5.3 The Galerkin (semidiscrete) solver

In this section we study the semidiscrete-in-space problem (5.1)–(5.2). We will study the operator that associates $g \mapsto (\lambda^h, u^h)$, by referring to its properties in the Laplace domain. We thus look at the problem of finding

$$\lambda^h \in X_h \qquad \text{such that} \qquad \langle \mu^h, \mathrm{V}(s)\lambda^h\rangle_\Gamma = \langle \mu^h, g\rangle_\Gamma \quad \forall \mu^h \in X_h. \tag{5.5}$$

and the constructing the potential

$$u^h = \mathrm{S}(s)\lambda^h. \tag{5.6}$$

We will then write

$$\lambda^h = \mathrm{G}^h_\lambda(s)g \qquad u^h = \mathrm{G}^h_u(s)g = \mathrm{S}(s)\mathrm{G}^h_\lambda(s)g.$$

What we mean with the expression 'for all h'. We want the analysis to be independent (as much as possible) of the particular choice of the discrete space X_h. We will use the convention of referring to a property being satisfied for all h to refer to the fact that the bound is completely independent of the choice of X_h. As we will see, all estimates will only use the fact that X_h is a closed subspace of $H^{-1/2}(\Gamma)$, but we will never use that it is finite dimensional. The results will therefore hold for $X_h = H^{-1/2}(\Gamma)$, in which case $\mathrm{G}^h_\lambda(s) = \mathrm{V}^{-1}(s)$.

Direct estimates. Using the fact that

$$\mathrm{Re}\Big(e^{\imath \,\mathrm{Arg}\, s}\langle \overline{\lambda}^h, \mathrm{V}(s)\lambda^h\rangle_\Gamma \geq C_\Gamma \frac{\sigma\underline{\sigma}}{|s|^2}\|\lambda^h\|^2_{-1/2,\Gamma} \qquad \forall \lambda^h \in X_h, \quad \forall s \in \mathbb{C}_+$$

(Proposition 2.6.1) we can bound

$$\|\mathrm{G}^h_\lambda(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}\| \leq C_\Gamma^{-1}\frac{|s|^2}{\sigma\underline{\sigma}} \qquad \forall s \in \mathbb{C}_+ \qquad \forall h. \tag{5.7}$$

Using the the estimate for the norm of $\mathrm{S}(s)$ (Proposition 2.6.2) we can bound

$$\|\mathrm{G}^h_u(s)\|_{H^{1/2}(\Gamma)\to H^1(\mathbb{R}^d)} \leq \|\mathrm{S}(s)\|_{H^{-1/2}(\Gamma)\to H^1(\mathbb{R}^d)}\|\mathrm{G}^h_\lambda(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \leq C_\Gamma\frac{|s|^3}{\sigma^2\underline{\sigma}^3}. \tag{5.8}$$

If we compare this result with the one for the continuous operator ($X_h = H^{-1/2}(\Gamma)$) given in Section 3.3, we can easily notice that the result is not optimal. We will next give a detailed proof of how to obtain a better estimate (that reproduces the one in Section 3.3) by using a very similar approach to the one used in the continuous case. We emphasize the importance of lowering the powers of $|s|$ and σ^{-1} in the bounds, both for mapping properties (Proposition 3.2.2) and for smoothness requirements to obtain convergence of the Convolution Quadrature method.

Analysis based on exotic transmission problems. We first need to introduce some notation. The polar set or annihilator of the set X_h is

$$X_h^\circ := \{\xi \in H^{1/2}(\Gamma) \,:\, \langle \lambda^h, \xi\rangle_\Gamma = 0 \quad \forall \lambda^h \in X_h\}.$$

A first observation is the fact that equation (5.5) is equivalent to

$$\lambda^h \in X_h \qquad \text{such that} \qquad \mathrm{V}(s)\lambda^h - g \in X_h^\circ. \tag{5.9}$$

The second observation is the equivalence

$$\lambda \in X_h \qquad \Longleftrightarrow \qquad \langle \lambda, \xi\rangle_\Gamma = 0 \quad \forall \xi \in X_h^\circ. \tag{5.10}$$

These two observations trigger the forthcoming analysis. Following [12], we will divide the process in four steps.

Proposition 5.3.1 (Step #1: transmission problem)**.** *Let $g \in H^{1/2}(\Gamma)$. Then $u^h = \mathrm{G}_u^h(s)g$ if and only if u^h is the solution of the following problem:*

$$u^h \in H^1(\mathbb{R}^d), \tag{5.11a}$$
$$\Delta u^h - s^2 u^h = 0 \quad \textit{in } \mathbb{R}^d \setminus \Gamma, \tag{5.11b}$$
$$\gamma u^h - g \in X_h^\circ, \tag{5.11c}$$
$$[\![\partial_\nu u^h]\!] \in X_h. \tag{5.11d}$$

Proof. If $u^h = \mathrm{G}_u^h(s)g = \mathrm{S}(s)\mathrm{G}_\lambda^h(s)g$, then it is clear that u^h satisfies (5.11a) and (5.11b). Moreover, defining $\lambda^h := \mathrm{G}_\lambda^h(s)g$, then

$$[\![\partial_\nu u^h]\!] = \lambda^h \in X_h \qquad \gamma u^h - g = \mathrm{V}(s)\lambda^h - g \in X_h^\circ,$$

by (5.5). This proves that u^h solves (5.11).

Reciprocally, if u^h satisfies (5.11), and we define $\lambda^h := [\![\partial_\nu u^h]\!] \in X_h$, then by the representation formula for the solutions of $\Delta - s^2$, we can write

$$u^h = \mathrm{S}(s)[\![\partial_\nu u^h]\!] - \mathrm{D}(s)[\![\gamma u^h]\!] = \mathrm{S}(s)\lambda^h$$

and therefore

$$\mathrm{V}(s)\lambda^h - g = \gamma u^h - g \in X_h^\circ,$$

which proves that $\lambda^h = \mathrm{G}_\lambda^h(s)g$ and $u^h = \mathrm{S}(s)\lambda^h = \mathrm{G}_u^h(s)g$. □

Proposition 5.3.2 (Step #2: variational formulation)**.** *Consider the space*

$$H_h := \{v^h \in H^1(\mathbb{R}^d) \ : \ \gamma v^h \in X_h^\circ\}.$$

The transmission problem (5.11) *is equivalent to the variational problem*

$$\left[\begin{array}{ll} u^h \in H^1(\mathbb{R}^d) & \gamma u^h - g \in X_h^\circ, \\ a_{s,\mathbb{R}^d}(u^h, v^h) = 0 & \forall v^h \in H_h. \end{array} \right. \tag{5.12}$$

Proof. If u^h solves (5.11), then for all $v^h \in H_h$

$$\begin{aligned} a_{s,\mathbb{R}^d}(u^h, v^h) &= (\nabla u^h, \nabla v^h)_{\mathbb{R}^d} + s^2(\Delta u^h, v^h)_{\mathbb{R}^d\setminus\Gamma} && (\Delta u^h = s^2 u^h) \\ &= \langle [\![\partial_\nu u^h]\!], \gamma v^h\rangle_\Gamma && (\text{definition of } [\![\partial_\nu \cdot]\!]) \\ &= 0. && ([\![\partial_\nu u^h]\!] \in X_h, \quad \gamma v^h \in X_h^\circ) \end{aligned}$$

Reciprocally, if u^h is a solution to (5.12) it follows readily that $\Delta u^h - s^2 u^h = 0$ in $\mathbb{R}^d \setminus \Gamma$, since $\mathcal{D}(\mathbb{R}^d \setminus \Gamma) \subset H_h$. Let now $\xi \in X_h^\circ$ and construct $v^h \in H^1(\mathbb{R}^d)$ such that $\gamma v^h = \xi$. It follows then that $v^h \in H_h$. Therefore

$$\begin{aligned} \langle [\![\partial_\nu u^h]\!], \xi\rangle_\Gamma &= \langle [\![\partial_\nu u^h]\!], \gamma v^h\rangle_\Gamma \\ &= (\nabla u^h, \nabla v^h)_{\mathbb{R}^d} + s^2(\Delta u^h, v^h)_{\mathbb{R}^d\setminus\Gamma} && (\text{definition of } [\![\partial_\nu \cdot]\!]) \\ &= a_{s,\mathbb{R}^d}(u^h, v^h) && (\Delta u^h = s^2 u^h) \\ &= 0 && (v^h \in H_h), \end{aligned}$$

and therefore $[\![\partial_\nu u^h]\!] \in X_h$ by (5.10). □

Proposition 5.3.3 (Step #3: energy/coercivity estimate)**.** *There exists C_Γ such that, for all $g \in H^{1/2}(\Gamma)$,*

$$\|u^h\|_{|s|,\mathbb{R}^d} \le C_\Gamma \frac{|s|^{3/2}}{\sigma\,\underline{\sigma}^{1/2}} \|g\|_{1/2,\Gamma}, \qquad \forall s \in \mathbb{C}_+, \quad \forall h, \tag{5.13}$$

where u^h is the solution of (5.12)*. Therefore*

$$\|\mathrm{G}_u^h(s)\|_{H^{1/2}(\Gamma)\to H^1(\mathbb{R}^d)} \le C_\Gamma \frac{|s|^{3/2}}{\sigma\,\underline{\sigma}^{3/2}}. \tag{5.14}$$

Proof. Recall first that the bilinear form $a_{s,\mathbb{R}^d}$ is coercive (see (2.13)) in the energy norm

$$\mathrm{Re}\big(e^{-\imath \mathrm{Arg}\, s} a_{s,\mathbb{R}^d}(v,\overline{v})\big) = \frac{\sigma}{|s|}\|v\|^2_{s,\mathbb{R}^d} \qquad \forall v \in H^1(\mathbb{R}^d). \tag{5.15}$$

Using the lifting of Proposition 2.5.1 it is easy to contruct liftings (right-inverses of the trace operator) $\gamma^+(s) : H^{1/2}(\Gamma) \to H^1(\mathbb{R}^d)$ such that

$$\|\gamma^+(s)g\|_{|s|,\mathbb{R}^d} \le C_1 \max\{1,|s|\}^{1/2}\|g\|_{1/2,\Gamma} \qquad \forall g \in H^{1/2}(\Gamma) \qquad \forall s \in \mathbb{C}_+. \tag{5.16}$$

Let us now test equations (5.12) with $v^h = u^h - u_g \in H_h$, where $u_g = \gamma^+(s)g$. Then

$$\begin{aligned}
\frac{\sigma}{|s|}\|u^h - u_g\|^2_{|s|,\mathbb{R}^d} &= \mathrm{Re}\big(e^{-\imath \mathrm{Arg}\, s} a_{s,\mathbb{R}^d}(u^h - u_g, \overline{u^h - u_g})\big) && \text{(by (5.15) or (2.13))}\\
&\le |a_{s,\mathbb{R}^d}(u_g, \overline{u^h - u_g})| && \text{(u^h solves (5.12))}\\
&\le \|u_g\|_{|s|,\mathbb{R}^d}\|u^h - u_g\|_{|s|,\mathbb{R}^d}. && \text{(by (2.12))}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u^h\|_{|s|,\mathbb{R}^d} &\le \|u_g\|_{|s|,\mathbb{R}^d} + \frac{|s|}{\sigma}\|u_g\|_{|s|,\mathbb{R}^d} \le 2\frac{|s|}{\sigma}\|u_g\|_{|s|,\mathbb{R}^d} && \text{(see above)}\\
&\le 2C_1 \frac{|s|}{\sigma}\max\{1,|s|\}^{1/2}\|g\|_{1/2,\Gamma} && \text{(by (5.16))}\\
&\le 2C_1 \frac{|s|^{3/2}}{\sigma\,\underline{\sigma}^{1/2}}\|g\|_{1/2,\Gamma}. && \text{(by (2.15))}
\end{aligned}$$

This proves (5.13). To prove (5.14) use Propositions 5.3.1 and 5.3.2 and the inequality $\underline{\sigma}\|u\|_{1,\mathbb{R}^d} \le \|u\|_{|s|,\mathbb{R}^d}$. Note that this entire group of proofs (from Proposition 5.3.1 and 5.3.3) just repeats at the discrete level what was proved at the continuous level in Section 3.3. □

Proposition 5.3.4 (Step #4: boundary wrap-up)**.** *There exists C_Γ such that*

$$\|\mathrm{G}_\lambda^h(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \le C_\Gamma \frac{|s|^2}{\sigma\underline{\sigma}} \qquad \forall s \in \mathbb{C}_+ \quad \forall h.$$

Proof. Proceeding in this four step program, this last step uses the fact that $\mathrm{G}_\lambda^h(s)g = [\![\partial_\nu \mathrm{G}_u^h(s)g]\!]$, Proposition 5.3.3 and Proposition 2.5.2, i.e., the fact that we can bound

$$\|[\![\partial_\nu u^h]\!]\|_{-1/2,\Gamma} \le C\frac{|s|^{1/2}}{\underline{\sigma}^{1/2}}\|u^h\|_{|s|,\mathbb{R}^d}.$$

Note that we obtain again the direct estimate given by coercivity in (5.7). This should not be a surprise since coercivity played a key role in the above proofs. □

5.4 The Galerkin error operator

Consider the problem of solving

$$\lambda^h \in X_h \qquad \text{such that} \qquad \langle \mu^h, \mathrm{V}(s)\lambda^h\rangle_\Gamma = \langle \mu^h, \mathrm{V}(s)\lambda\rangle_\Gamma \quad \forall \mu^h \in X_h$$

and then defining $u^h = \mathrm{S}(s)\lambda^h$. We want to study the following operators

$$\mathrm{E}^h_\lambda(s)\lambda := \lambda^h - \lambda, \qquad \mathrm{E}^h_u(s)\lambda := u^h - u = \mathrm{S}(s)\mathrm{E}^h_\lambda(s)\lambda.$$

It is clear that

$$\mathrm{E}^h_\lambda(s) = \mathrm{G}^h_\lambda(s)\mathrm{V}(s) - \mathrm{I}, \qquad \mathrm{E}^h_u(s) = \mathrm{G}^h_u(s)\mathrm{V}(s) - \mathrm{S}(s) = \mathrm{S}(s)(\mathrm{G}^h_\lambda(s)\mathrm{V}(s) - \mathrm{I})$$

and that

$$\mathrm{E}^h_\lambda(s) = [\![\partial_\nu \cdot]\!]\mathrm{E}^h_u(s).$$

The process is very similar to the one developed in Section 5.3, and we will only sketch the four step program in the proof. Details are left to the reader. We will use agan the space

$$H_h := \{v^h \in H^1(\mathbb{R}^d) \,:\, \gamma v^h \in X_h^\circ\}.$$

Note the parallelism of the transmission problems (5.11) and (5.20). They are based on the same set of exotic transmission conditions. While the study of the Galerkin solver leads to a non-homogeneous essential transmission condition, the Galerkin error operator takes us to a non-homogeneous natural transmission condition.

Proposition 5.4.1 (Step #1: transmission problem)**.** *Let $\lambda \in H^{-1/2}(\Gamma)$. Then $\varepsilon^h_u = \mathrm{E}^h_u(s)\lambda$ if and only if*

$$\varepsilon^h_u \in H^1(\mathbb{R}^d), \tag{5.17a}$$

$$\Delta\varepsilon^h_u - s^2\varepsilon^h_u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{5.17b}$$

$$\gamma\varepsilon^h_u \in X_h^\circ, \tag{5.17c}$$

$$[\![\partial_\nu \varepsilon^h_u]\!] + \lambda \in X_h. \tag{5.17d}$$

Proposition 5.4.2 (Step #2: variational formulation)**.** *The transmission problem* (5.20) *is equivalent to the variational problem*

$$\left[\begin{array}{l} \varepsilon^h_u \in H_h, \\ a_{s,\mathbb{R}^d}(\varepsilon^h_u, v^h) = -\langle \lambda, \gamma v^h\rangle_\Gamma \qquad \forall v^h \in H_h. \end{array}\right. \tag{5.18}$$

Proposition 5.4.3 (Step #3: energy/coercivity estimate)**.** *There exists C_Γ such that, for all $\lambda \in H^{-1/2}(\Gamma)$,*

$$|\!|\!|\varepsilon^h_u|\!|\!|_{|s|,\mathbb{R}^d} \le C_\Gamma \frac{|s|}{\sigma\,\underline{\sigma}} \|\lambda\|_{1/2,\Gamma}, \qquad \forall s \in \mathbb{C}_+, \quad \forall h,$$

where ε^h_u is the solution of (5.21)*. Therefore*

$$\|\mathrm{E}^h_u(s)\|_{H^{-1/2}(\Gamma)\to H^1(\mathbb{R}^d)} \le C_\Gamma \frac{|s|}{\sigma\,\underline{\sigma}^2}. \tag{5.19}$$

Proposition 5.4.4 (Step #4: boundary wrap-up)**.** *There exists C_Γ such that*

$$\|\mathrm{E}^h_\lambda(s)\|_{H^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \le C_\Gamma \frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}} \qquad \forall s \in \mathbb{C}_+ \quad \forall h.$$

Galerkin semidiscretization error. Let finally $\Pi^h : H^{-1/2}(\Gamma) \to X_h$ be the orthogonal projection operator onto X_h. Note that

$$\mathrm{E}_\lambda^h(s)\Pi^h = 0,$$

as a consequence of the fact that if $\lambda \in X_h$, then $\lambda^h = \lambda$. In other words this reflects the fact that Π^h and $\mathrm{E}_\lambda^h(s)$ are complementary projections. We will use this property in the following equivalent form

$$\mathrm{E}_\lambda^h(s) = \mathrm{E}_\lambda^h(s)(\mathrm{I} - \Pi^h) \qquad \mathrm{E}_u^h(s) = \mathrm{E}_u^h(s)(\mathrm{I} - \Pi^h),$$

which is a higly compacted form of a Céa estimate.

5.5 Error estimates

Review. Let us briefly recall the fully discrete method destined to approximate the solution of

$$\mathcal{V} * \lambda = g \qquad u = \mathcal{S} * \lambda. \tag{5.20}$$

The Galerkin solver, defined in the Laplace domain in Section 5.3, corresponds to convolution with the operator

$$\mathcal{G}_\lambda^h := \mathcal{L}^{-1}\{\mathrm{G}_\lambda^h\}.$$

The semidiscrete system is then

$$\lambda^h = \mathcal{G}_\lambda^h * g \qquad u^h = \mathcal{S} * \lambda^h, \tag{5.21}$$

while the full discrete CQ-BEM scheme consists of

$$\lambda_\kappa^h = \mathcal{G}_{\lambda,\kappa}^h * g \qquad u_\kappa^h = \mathcal{S}_\kappa * \lambda_\kappa^h. \tag{5.22}$$

Note that CQ has been applied to both convolution processes in (5.21), i.e., to the semidiscrete version of the equation $\mathcal{V} * \lambda = g$ and to the potential posprocessing $u = \mathcal{S} * \lambda^h$. The Laplace domain version of (5.22) is

$$\Lambda_\kappa^h(s) = \mathrm{G}_{\lambda,\kappa}^h(s)\mathrm{G}(s) \qquad \mathrm{U}_\kappa^h(s) = \mathrm{S}_\kappa(s)\Lambda_\kappa^h(s) = \mathrm{G}_{u,\kappa}^h(s)\mathrm{G}(s),$$

or equivalently,

$$\Lambda_\kappa^h(s) = \mathrm{G}_\lambda^h(s_\kappa)\mathrm{G}(s), \qquad \mathrm{U}_\kappa^h(s) = \mathrm{G}_u^h(s_\kappa)\mathrm{G}(s), \quad \text{where} \quad s_\kappa = \kappa^{-1}p(e^{-s\kappa}).$$

Galerkin semidiscretization error. Let finally

$$\mathcal{E}_\lambda^h := \mathcal{L}^{-1}\{\mathrm{E}_\lambda^h\} \qquad \mathcal{E}_u^h := \mathcal{L}^{-1}\{\mathrm{E}_u^h\}.$$

Then

$$\lambda^h - \lambda = \mathcal{E}_\lambda^h * \lambda = \mathcal{E}_\lambda^h * (\lambda - \Pi^h\lambda)$$

and

$$u^h - u = \mathcal{S} * (\lambda^h - \lambda) = \mathcal{E}_u^h * \lambda = \mathcal{E}_u^h * (\lambda - \Pi^h\lambda).$$

Proposition 3.2.2 applied to the convolution operators with symbols E_λ^h and E_u^h (analysed in Propositions 5.4.3 and 5.4.4) gives a closed analysis of the semidiscretization error in terms of approximation errors.

Convolution Quadrature error. Since we have been careful in showing how the constants that bound the Galerkin solvers in the Laplace domain are independent of h (see Propositions 5.3.3 and 5.3.4), the analysis of the time discretization is now just a case of using the Convolution Quadrature estimates (Proposition 4.6.1 and 4.6.2) to bound

$$\|(\lambda^h - \lambda^h_\kappa)(t)\|_{-1/2,\Gamma} = \|(\mathcal{G}^h_\lambda * g - \mathcal{G}^h_{\lambda,\kappa} * g)(t)\|_{-1/2,\Gamma}$$

and

$$\|(u^h - u^h_\kappa)(t)\|_{1,\mathrm{R}^d} = \|(\mathcal{G}^h_u * g - \mathcal{G}^h_{u,\kappa} * g)(t)\|_{1/2,\Gamma}.$$

All these results are collected in the following propositions. Note that regularity requirements for full order of convergence of the density and of the potential are different. We will come back and improve all these results when we approach the analysis of potentials (continuous and semidiscrete) and of the fully discrete methods using time domain techniques.

5.6 Exercises

1. (Section 5.3) Prove (5.10). You can actually prove the samel statement in this more general setting: X_h is a closed subspace of a Hilbert space X and $X_h^\circ \subset X'$ is its polar set. (**Hint.** You can decompose $X = X_h \oplus X_h^\perp$ and relate $X_h^\top \subset X$ with $X_h^\circ \subset X'$ with the Riesz representation map.)

2. (Section 5.4) Prove all the results of this section.

3. (Section 5.4) Show that the Galerkin error operator $\mathrm{E}^h_\lambda(s)$ is a projection. Give a characterization of its range. (**Hint.** The range depends on s.)

Chapter 6

Second order equations by separation of variables

In this chapter we are going to introduce some elementary tools of evolution equations and to apply them for some further analysis of time domain potentials and integral operators. Most of the results in the abstract treatment of evolution equations that follow can be obtained with elementary tools of the theory of strongly continuous semigroups of operators (of groups of isometries actually). We are going to approach this theory with a different point of view, namely the method of separation of variables. (By the way, this approach is more restrictive, but it completely serves our purposes.) The main tools for the analysis are the Hilbert-Schmidt theorem and some basic results on continuity of functions defined by series. They will be reviewed in the last section of this chapter for ease of reference. The results of this chapter are adapted and expanded from [9, Section 8] and [20, Section 3].

6.1 The basic setup

Two spaces to start with. We consider two real Hilbert spaces

$$V \subset H$$

with **continuous, compact and dense injection.** The norm of H, $\|\cdot\|_H$ will be the one we will use to measure kinetic energy. We also consider a finite dimensional space

$$M \subset V$$

that we will call the space of **rigid motions,** the orthogonal projection $P : H \to M$ and the closed subspace

$$H_0 := \{u \in H \,:\, (u,m)_H = 0 \quad \forall m \in M\} = M^\perp.$$

On V we assume that the norm is given in the form

$$\|v\|_V^2 := [v,v] + (Pv,v)_H = [v,v] + \|Pv\|_H^2$$

where $[\,\cdot\,,\,\cdot\,]$ is a symmetric positive semidefinite bilinear form such that

$$[\,\cdot\,, m] = 0 \quad \forall m \in M.$$

This property is equivalent to

$$[m, m] = 0 \quad \forall m \in M.$$

Note that since $\|\,\cdot\,\|_V$ is a norm, it is easy to prove that

$$[u, v] = 0 \quad \forall v \in V \qquad \Longleftrightarrow \qquad u \in M.$$

Note finally that for an element of the space

$$V_0 := \{u \in V \,:\, Pu = 0\} = V \cap H_0$$

the norm is just $\|u\|_V = [u, u]^{1/2}$. This seminorm in V will be the one taking care of measuring **potential energy**. The continuity of the injection of V into H can be expressed with the inequality

$$\|u\|_H \le C_\circ \|u\|_V \qquad \forall u \in V. \tag{6.1}$$

For reasons that will be understood in the following chapters, it is important to keep track of all occurences of this constant $C_\circ$.

A third space and an operator. We assume the existence of a third space

$$D(A) \subset V \subset H \qquad \text{with} \qquad M \subset D(A)$$

and a linear operator $A : D(A) \to H$ such that the norm

$$\|u\|^2_{D(A)} := \|Au\|^2_H + \|u\|^2_V$$

makes $D(A)$ a Hilbert space. The final elements for this abstract construction are:

(a) an **abstract Green's Identity**

$$(Au, v)_H + [u, v] = 0 \qquad \forall u \in D(A) \quad \forall v \in V, \tag{6.2}$$

(b) and a **surjectivity property**:

$$A - I : D(A) \to H \quad \text{is onto.}$$

Proposition 6.1.1. *Assuming all the hypotheses above, it follows that:*

(a) $\ker A = M$

(b) $R(A) := \{Au \,:\, u \in D(A)\} \subset H_0$

(c) $A - I : D(A) \to H$ *is invertible*

(d) $A - I : M \to M$ *is just the minus identity operator.*

(e) $A - I : D(A) \cap H_0 \to H_0$ *is invertible.*

Proof. To prove (a) note first that if $Au = 0$, then by (6.2) it follows that $[u, u] = 0$ and therefore $u \in M$. Conversely, if $u \in M$, then

$$0 = (Au, v)_H + [u, v] = (Au, v)_H \quad \forall v \in V.$$

This implies that $Au = 0$ because V is dense in H. To prove (b) note that if $f = -Au$, then

$$(f, m)_H = (-Au, m)_H = [u, m] = 0 \quad \forall m \in M.$$

To prove (c) we only need to show that $A - I$ is injective. If $-Au + u = 0$, then, using (6.2),

$$0 = -(Au, u)_H + (u, u)_H = [u, u] + (u, u)_H$$

and therefore $u = 0$. Properties (d) and (e) are straightforward. □

Removing density as a hypothesis. The density of the inclusion of V in H is only used in proving that $M \subset \ker A$ (we will use it later on). Assuming that $M \subset \ker A$, we can actually get to prove that V is dense in H as a byproduct of a much more demanding theoretical development.

A fast list of examples. It is useful to keep in mind many of the following examples to see how this abstract framework will help to describe several problems related to wave propagation. In all the examples Ω is a bounded set with Lipschitz boundary and the norm of $L^2(\Omega)$ is the standard one.

(1) Take

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad D(A) = \{u \in H_0^1(\Omega) \,:\, \Delta u \in L^2(\Omega)\},$$

with the Dirichlet form as the bilinear form

$$[u, v] := (\nabla u, \nabla v)_\Omega,$$

with $A := \Delta$. In this case $M = \{0\}$ and the constant $C_\circ$ in (6.1) is that of the Poincaré-Friedrichs inequality.

(2) We can also take

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad D(A) := \{u \in H^1(\Omega) \,:\, \Delta u \in L^2(\Omega), \quad \partial_\nu u = 0\}$$

with the same $[\,\cdot\,,\,\cdot\,]$, $A := \Delta$ again and $M = \mathbb{P}_0(\Omega)$, the space of constant functions. The abstract Green's Identity is these two first examples is just a consequence of Green's identity after substituting the corresponding homogeneous boundary condition.

(3) Related to the biharmonic operator we can take

$$H = L^2(\Omega), \quad V = H_0^2(\Omega), \quad D(A) := \{u \in H_0^2(\Omega) \,:\, \Delta^2 u \in L^2(\Omega)\}$$

with the bilinear form

$$[u, v] := (\Delta u, \Delta v)_\Omega,$$

the space of rigid motions $M := \mathbb{P}_1(\Omega)$, including all plane displacements and $A := -\Delta^2$. The abstract Green's Identity is now a weak formulation of the Rayleigh–Green theorem. The bilinear form can also be written as follows

$$[u, v] := \nu(\Delta u, \Delta v)_\Omega + (1 - \nu)(\mathrm{H}\, u, \mathrm{H}\, v)_\Omega, \tag{6.3}$$

where $\mathrm{H}\, u$ is the Hessian matrix of u and $\nu \in [0, 1]$. The fact that these two bilinear forms coincide in $H_0^2(\Omega)$ follows from a simple density argument.

(4) The bilinear form (6.3) and the operator $A := -\Delta^2$ can be used for three choices of the space V

$$H^2(\Omega) \cap H_0^1(\Omega), \qquad H^2(\Omega), \quad \text{or} \quad \{u \in H^2(\Omega) : \partial_\nu u = 0\}.$$

In the case of smooth boundary $\partial\Omega$, the corresponding space $D(A)$ can be described using two higher order boundary conditions. (In the general case, unexpected difficulties are met because of paradoxes related to the impossibility of cleanly separating the boundary conditions γu and $\partial_\nu u$ for elements of $H^2(\Omega)$ in some domains.)

(5) The linear elasticity system fits in this frame by using the bilinear form

$$[\mathbf{u}, \mathbf{v}] := \big(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})\big)_\Omega \qquad \boldsymbol{\varepsilon}(\mathbf{u}) := \tfrac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^\top)$$

where $\mathcal{C}$ is a linear operator (a tensor) allowing for the bilinear form to be symmetric and the coercivity condition

$$[\mathbf{u}, \mathbf{u}] \geq C_{\mathcal{C}} \|\boldsymbol{\varepsilon}(\mathbf{u})\|_\Omega^2 \qquad \forall \mathbf{u} \in H^1(\Omega)^d$$

to hold. With the spaces

$$H := L^2(\Omega)^2, \qquad V := H_0^1(\Omega)^d, \qquad D(A) := \{\mathbf{u} \in V \,:\, \mathrm{div}\,(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) \in L^2(\Omega)^d\},$$

$M = \{0\}$ and $A := \mathrm{div}\,(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}))$ we can describe problems where displacement is given on the boundary. The weak form of the normal stress can be defined as an element of $H^{-1/2}(\Gamma)^d$ with Betti's formula (Green's formula for the elasticity system):

$$\langle \boldsymbol{\sigma}_\nu(\mathbf{u}), \gamma \mathbf{v} \rangle_\Gamma := (\mathrm{div}\,(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})), \mathbf{v})_\Omega + (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega$$

With the same H but

$$V := H^1(\Omega)^d, \qquad D(A) := \{\mathbf{u} \in V \,:\, \mathrm{div}\,(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) \in L^2(\Omega)^d, \quad \boldsymbol{\sigma}_\nu(\mathbf{u}) = \mathbf{0}\},$$

we can describe problem with free boundary conditions. In this case, there is a space of rigid motions related to the problem: in two dimensions, M contains the functions

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 & -a_3 \\ a_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

whereas in three dimensions, it contains the functions

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 0 & a_4 & -a_5 \\ -a_4 & 0 & a_6 \\ a_5 & -a_6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Fast forward. The remainder of this chapter is related to proving existence, uniqueness and stability properties for some initial value problems related to second order differential equations and the operator A. For the reader who is not interested in learning about the proofs (or is already acquainted with deeper mathematical techniques from which these result follow), we give here the results in a fast way.

- The Cauchy problem: if $u_0 \in D(A)$ and $v_0 \in V$, then the problem

  $$\ddot{u}(t) = Au(t) \quad t \geq 0, \qquad u(0) = u_0, \qquad \dot{u}(0) = v_0$$

  has a unique solution in the space

  $$\mathcal{C}^2([0,\infty); H) \cap \mathcal{C}^1([0,\infty); V) \cap \mathcal{C}([0,\infty); D(A)).$$

  Existence of solution is proved in Proposition 6.3.1. Uniqueness is proved in Proposition 6.3.2.

- Strong solutions of non-homogeneous problems: if $f : [0,\infty) \to V_0$ is continuous, the problem

  $$\ddot{u}(t) = Au(t) + f(t) \quad t \geq 0, \qquad u(0) = \dot{u}(0) = 0$$

  has a unique solution in the space

  $$\mathcal{C}^2([0,\infty); H) \cap \mathcal{C}^1([0,\infty); V) \cap \mathcal{C}([0,\infty); D(A)).$$

  Moreover, for all $t \geq 0$

  $$C_\circ^{-1}\|u(t)\|_H \leq \|u(t)\|_V \leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \qquad \|Au(t)\|_H \leq \int_0^t \|f(\tau)\|_V \mathrm{d}\tau.$$

  Existence of solution is shown in Proposition 6.4.2. The bounds are given in Proposition 6.4.3. We will also provide the following bounds for the derivative

  $$\|\dot{u}(t)\|_H \leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \qquad \|\dot{u}(t)\|_V \leq \int_0^t \|f(\tau)\|_V \mathrm{d}\tau.$$

- Weak solutions of non-homogeneous problems: if $f : [0,\infty) \to H_0$ is continuous, the problem

$$\langle \ddot{u}(t), v\rangle_{V'\times V} + [u(t), v] = (f(t), v)_H \quad \forall v \in V, \quad \forall t \geq 0, \qquad u(0) = \dot{u}(0) = 0$$

has a unique solution in the space

$$\mathcal{C}^2([0,\infty); V') \cap \mathcal{C}^1([0,\infty); H) \cap \mathcal{C}([0,\infty); V).$$

Moreover, for all $t \geq 0$

$$C_\circ^{-1}\|u(t)\|_H \leq \|u(t)\|_V \leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau \quad \text{and} \quad \|\dot{u}(t)\|_H \leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau.$$

This is proved in Proposition 6.5.1.

- Finally, in Proposition 6.5.2 we will show that for $f \in \mathcal{C}^1([0,\infty); H_0)$ with $f(0) = 0$, the problem

$$\ddot{u}(t) = Au(t) + f(t) \quad t \geq 0, \qquad u(0) = \dot{u}(0) = 0$$

has a unique solution in the space

$$\mathcal{C}^2([0,\infty); H) \cap \mathcal{C}^1([0,\infty); V) \cap \mathcal{C}([0,\infty); D(A))$$

satisfying

$$C_\circ^{-1}\|u(t)\|_H \leq \|u(t)\|_V \leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \qquad \|Au(t)\|_H \leq 2\int_0^t \|\dot{f}(\tau)\|_H \mathrm{d}\tau.$$

The theory about the Cauchy problem is standard in the theory of C_0-groups of isometries generated by unbounded operators. The theory for non-homogeneous problems with the given regularity for f is not so standard.

6.2 Green's operator

Definition of the abstract Green operator. We return to the abstract setting of Section 6.1 and we define the operator $G : H_0 \to H_0$ by

$$Gf = u \qquad \text{where} \qquad \left[\begin{array}{l} u \in V_0 \\ [u, v] = (f, v)_H \quad \forall v \in V \quad \big(\Leftrightarrow \forall v \in V_0\big). \end{array}\right.$$

This operator is well defined because it is just the Riesz-Fréchet representation in V_0 of the bounded linear functional $(f, \,\cdot\,)_H : V_0 \to \mathbb{R}$. Note that if $u \in V_0 \cap D(A)$ and $-Au = f$, then by the abstract Green Identity

$$0 = (Au, v)_H + [u, v] = -(f, v)_H + [u, v] \qquad \forall v \in V,$$

and therefore $u = Gf$. This means that G is providing us a solution of the equation $-Au = f$. The following properties are simple consequences of the hypotheses for the spaces and the operators.

Proposition 6.2.1 (Functional properties of Green's operator)**.** *Assuming all the hypotheses of Section 6.1, it follows that:*

(a) *With the constant* $C_\circ > 0$ *of inequality* (6.1)

$$\|Gf\|_V \leq C_\circ \|f\|_H \qquad \forall f \in H_0. \tag{6.4}$$

(b) *The operator* G *is compact.*

(c) *The operator* G *is self-adjoint and positive definite.*

Proof. To prove (a) we just need to notice that since $u = Gf \in V_0$, then

$$\|u\|_V^2 = [u,u] = (f,u)_H \leq \|f\|_H \|u\|_H \leq C_\circ \|f\|_H \|u\|_V.$$

Since the injection of V_0 in H_0 is compact, the bound (6.4) proves the compactness of $G : H_0 \to H_0$. The proof of (c) is a direct consequence of the definition:

$$(f, Gg)_H = [Gf, Gg] = [Gg, Gf] = (g, Gf)_H.$$

From this equality it also follows that G is positive semidefinite. Positive definiteness is then reduced to showing that G is injective: if $Gf = 0$ then

$$0 = [Gf, v] = (f, v)_H = 0 \qquad \forall v \in V,$$

but since V is dense in H it follows that $f = 0$. $\square$

The associated spectral decomposition. Proposition 6.2.1 leaves us in place to apply the Hilbert-Schmidt decomposition theorem (Theorem 6.7.1 below): we can find a Hilbert basis of H_0, $\{\phi_n\}$, and a sequence of positive non-increasing eigenvalues $\lambda_n > 0$ converging to zero, and write

$$G = \sum_{n=1}^{\infty} \lambda_n (\,\cdot\,, \phi_n)_H \phi_n. \tag{6.5}$$

Every $u \in H$ can be expressed as an H-orthogonal sum.

$$u = Pu + \sum_{n=1}^{\infty} (u, \phi_n)_H \phi_n. \tag{6.6}$$

As a first observation, note that, since $G\phi_n = \lambda_n \phi_n$, then

$$[\phi_n, \phi_m] = \lambda_n^{-1} [G\phi_n, \phi_m] = \lambda_n^{-1} (\phi_n, \phi_m)_H = \lambda_n^{-1} \delta_{n,m} \qquad \forall n, m \tag{6.7}$$

and that therefore

$$\lambda_n = \frac{(\phi_n, \phi_n)_H}{[\phi_n, \phi_n]} = \frac{\|\phi_n\|_H^2}{\|\phi_n\|_V^2} \leq C_\circ^2, \tag{6.8}$$

following (6.1). We now move on to characterize the spaces V and $D(A)$ and the operator A in terms of the elements (eigenvalues and eigenfunctions) that appear in the series representation (6.5) of Green's operator.

Proposition 6.2.2 (Series characterization of the energy space)**.** *Assuming the hypotheses of Section 6.1 and with the spectral decomposition of G given in* (6.5) *it follows that:*

(a) $\{\lambda_n^{1/2}\phi_n\}$ *is a Hilbert basis of* V_0.

(b) *The seminorm of V (the norm of V_0) can be represented in series form as*

$$[u, u] = \sum_{n=1}^{\infty} \lambda_n^{-1} |(u, \phi_n)_H|^2 \qquad \forall u \in V.$$

Therefore

$$\|u\|_V^2 = \sum_{n=1}^{\infty} \lambda_n^{-1} |(u, \phi_n)_H|^2 + \|Pu\|_H^2 \qquad \forall u \in V.$$

(c) $V_0 = R(G^{1/2})$, *where*

$$G^{1/2} = \sum_{n=1}^{\infty} \lambda_n^{1/2} (\,\cdot\,, \phi_n)_H \phi_n.$$

Consequently, $V = M \oplus R(G^{1/2})$.

Proof. The identities (6.7) prove that $\lambda_n^{1/2}\phi_n$ is an orthonormal set in V_0. Since

$$[u, \phi_n] = \lambda_n^{-1} (\phi_n, u)_H \qquad \forall n \quad \forall u \in V_0, \tag{6.9}$$

it follows that $\{\phi_n\}$ is complete in V_0 and (a) is proved. In turn, this shows that

$$[u, u] = \sum_{n=1}^{\infty} \big|[u, \lambda_n^{1/2}\phi_n]\big|^2 = \sum_{n=1}^{\infty} \lambda_n \big|[u, \phi_n]\big|^2 = \sum_{n=1}^{\infty} \lambda_n^{-1} |(u, \phi_n)_H|^2 \quad \forall u \in V_0$$

(we have applied (6.9) again), from where (b) follows readily. Finally, by Picard's criterion (Theorem 6.7.2 below) applied to the compact self-adjoint positive definite operator $G^{1/2}$ (see Section 6.7) it follows that

$$R(G^{1/2}) = \{u \in H_0 \,:\, \sum_{n=1}^{\infty} \lambda_n^{-1} |(u, \phi_n)_H|^2 < \infty\}.$$

Therefore $V_0 \subset R(G^{1/2})$ and $[u, u]^{1/2} = \|u\|_V = \|u\|_{R(G^{1/2})}$ for all $u \in V_0$. However, the linear combinations of the eigenfunctions $\{\phi_n\}$ form a dense subset of V_0 that is also dense in $R(G^{1/2})$, which implies the equality of both sets. □

Proposition 6.2.3 (Series characterization of $D(A)$)**.** *Assuming the hypotheses of Section 6.1 and with the spectral decomposition of G given in* (6.5) *it follows that*

$$R(G) = \{u \in H_0 \,:\, \sum_{n=1}^{\infty} \lambda_n^{-2} |(u, \phi_n)_H|^2 < \infty\} = D(A) \cap H_0 \tag{6.10}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n^{-2}|(u,\phi_n)_H|^2+\|Pu\|_H^2 &\leq \|u\|_{D(A)}^2\\ &\leq (1+C_\circ^2)\sum_{n=1}^{\infty}\lambda_n^{-2}|(u,\phi_n)_H|^2+\|Pu\|_H^2 \quad \forall u\in D(A).\end{aligned}$$

Proof. Let start by defining $B : R(G) \to H_0$ by

$$Bu := -\sum_{n=1}^{\infty}\lambda_n^{-1}(u,\phi_n)_H\phi_n$$

(i.e., $B = -G^{-1}$).

If $u \in D(A)\cap H_0 \subset V_0$, then by Green's Identity (6.2)

$$[u,v] = -(Au,v)_H \qquad \forall v\in V$$

and therefore, by the definition of G, $u = -GAu \in R(G)$. This also implies that

$$\begin{aligned}(u,\phi_n)_H &= -(GAu,\phi_h)_H\\ &= -(Au,G\phi_n)_H \qquad (G \text{ is selfadjoint})\\ &= -\lambda_n(Au,\phi_n)_H \qquad (G\phi_n=\lambda_n\phi_n)\end{aligned}$$

and therefore

$$\begin{aligned}Bu &= -\sum_{n=1}^{\infty}\lambda_n^{-1}(u,\phi_n)_H\phi_n\\ &= \sum_{n=1}^{\infty}(Au,\phi_n)_H\phi_n\\ &= Au. \qquad\qquad (Au\in H_0 \text{ by Proposition 6.1.1(b)})\end{aligned}$$

We have so far proved that $D(A)\cap H_0 \subset R(G)$ and that $Au = Bu$ for all $u \in D(A)\cap H_0$.

We already know by Proposition 6.1.1(e) that $A - I : D(A)\cap H_0 \to H_0$ is a bijection. Using the characterization of $R(G)$ given by Picard's criterion (Theorem 6.7.2) we can easily show that $I - B : R(G) \to H_0$ is also a bijection. However, $(I-B)|_{D(A)\cap H_0} = I - A$, and therefore $D(A)\cap H_0 = R(G)$ and

$$Au = -\sum_{n=1}^{\infty}\lambda_n^{-1}(u,\phi_n)_H\phi_n \qquad \forall u\in D(A). \tag{6.11}$$

(Note that the expression (6.11) includes the fact that $Au = 0$ for $u \in M$.) Moreover, if $u \in D(A)\cap H_0 = R(G)$, then

$$\|u\|_{D(A)}^2 = \sum_{n=1}^{\infty}\lambda_n^{-2}|(u,\phi_n)_H|^2+[u,u] = \sum_{n=1}^{\infty}(\lambda_n^{-2}+\lambda_n^{-1})|(u,\phi_n)_H|^2 \tag{6.12}$$

by Proposition 6.2.2(b) and therefore

$$\|u\|^2_{D(A)} = \sum_{n=1}^{\infty}(\lambda_n^{-2} + \lambda_n^{-1})|(u, \phi_n)_H|^2 + \|Pu\|^2_H \quad \forall u \in D(A). \tag{6.13}$$

The final bound is a simple consequence of (6.13) and (6.8). □

Final remark. As a direct consequence of (6.11) it is clear that

$$-A\phi_n = \lambda_n^{-1}\phi_n \qquad \forall n.$$

Moreover, if we look for eigenvalues of A, apart from the inverses of the eigenvalues of the Green operator, the only other possible eigenvalue is $\lambda = 0$ with the elements of the space of rigid motions M as eigenfunctions. Therefore, the eigenvalues of the (unbounded) operator $-A$ diverge to infinity.

6.3 The Cauchy problem

Abstract wave equations. We still assume all the hypotheses of Section 6.1 and consider the problem

$$\left[\begin{array}{ll} \ddot{u}(t) = Au(t) & t \in [0, \infty), \\ u(0) = u_0, & \\ \dot{u}(0) = v_0, & \end{array}\right. \tag{6.14}$$

for initial data $u_0 \in D(A)$ and $v_0 \in V$. Associated to the spectral series representation of the unbounded operator A

$$Au = -\sum_{n=1}^{\infty} \lambda_n^{-1}(u, \phi_n)_H \phi_n,$$

we can define the natural frequencies (modes):

$$\xi_n := \lambda_n^{-1/2} > 0 \qquad \lim_{n\to\infty} \xi_n = \infty.$$

Proposition 6.3.1 (Solution by separation of variables)**.** *Let $u_0 \in D(A)$ and $v_0 \in V$. The function*

$$u(t) := Pu_0 + t\,Pv_0 + \sum_{n=1}^{\infty} \cos(\xi_n\, t)(u_0, \phi_n)_H \phi_n + \sum_{n=1}^{\infty} \xi_n^{-1} \sin(\xi_n\, t)(v_0, \phi_n)_H \phi_n \tag{6.15}$$

is a strong solution of the Cauchy problem (6.14)

$$u \in \mathcal{C}^2(\mathbb{R}; H) \cap \mathcal{C}^1(\mathbb{R}; V) \cap \mathcal{C}(\mathbb{R}; D(A)). \tag{6.16}$$

Proof. Consider first the functions

$$c_n(t) := \cos(\xi_n\, t)(u_0, \phi_n)_H \phi_n, \qquad c_n : \mathbb{R} \to D(A).$$

By (6.13), it follows that

$$\|c_n(t)\|^2_{D(A)} = (\lambda_n^{-2} + \lambda_n^{-1})|\cos(\xi_n\, t)|^2 |(u_0, \phi_n)_H|^2 \le (1 + C_\circ^2)\lambda_n^{-2}|(u_0, \phi_n)_H|^2$$

with

$$\sum_{n=1}^{\infty} \lambda_n^{-2} |(u_0, \phi_n)_H|^2 < \infty \tag{6.17}$$

because $u_0 \in D(A)$, by Proposition 6.2.3. The values $c_n(t)$ and $c_m(t)$ are $D(A)$-orthogonal for all t. We can then apply Lemma 6.7.3 to show that

$$c(t) := Pu_0 + \sum_{n=1}^{\infty} \cos(\xi_n\, t)(u_0, \phi_n)_H \phi_n \tag{6.18}$$

defines a continuous function $\mathbb{R} \to D(A)$. Its value at $t = 0$ is $c(0) = u_0$ by (6.6). This function is also continuous from $\mathbb{R}$ to V. We next note that

$$\dot{c}_n(t) = -\xi_n \sin(\xi_n\, t)(u_0, \phi_n)_H \phi_n = -\lambda_n^{-1/2} \sin(\xi_n\, t)(u_0, \phi_n)_H \phi_n$$

and that by Proposition 6.2.2(b)

$$\|\dot{c}_n(t)\|_V^2 = \lambda_n^{-1} \big|\lambda_n^{-1/2} \sin(\xi_n\, t)(u_0, \phi_n)_H\big|^2 \le \lambda_n^{-2} |(u_0, \phi_n)_H|^2.$$

Using Lemma 6.7.4, we deduce that the function $c(t)$ in (6.18) is $\mathcal{C}^1$ with values in V (and therefore in H). Its derivative is

$$\dot{c}(t) = -\sum_{n=1}^{\infty} \lambda_n^{-1/2} \sin(\xi_n\, t)(u_0, \phi_n)_H \phi_n$$

with convergence in V (and in H). Also $\dot{c}(0) = 0$. We can proceed similarly to show that $\dot{c} \in \mathcal{C}^1(\mathbb{R}, H)$ and that

$$\ddot{c}(t) = -\sum_{n=1}^{\infty} \xi_n^2 (u_0, \phi_n)_H \phi_n = -\sum_{n=1}^{\infty} \lambda_n^{-1} (u_0, \phi_n)_H \phi_n = A\big(u(t)\big).$$

We can use exactly the same ideas to analyze

$$s(t) := t\, Pv_0 + \sum_{n=1}^{\infty} \xi_n^{-1} \sin(\xi_n\, t)(v_0, \phi_n)_H \phi_n = t\, Pv_0 + \sum_{n=1}^{\infty} s_n(t),$$

using now the bound

$$\sum_{n=1}^{\infty} \lambda_n^{-2} \big|\xi_n^{-1}(v_0, \phi_n)_H\big|^2 = \sum_{n=1}^{\infty} \lambda_n^{-1} |(v_0, \phi_n)_H|^2$$

due to the fact that $v_0 \in V$ and Proposition 6.2.2. $\square$

Going backwards in time. It is clear that the expression in (6.15) works also for negative values of t and that we have found a formula for the solution of (6.14) valid for all $t \in \mathbb{R}$.

Proposition 6.3.2 (Energy conservation). *Let u be the solution of the Cauchy problem* (6.14) *given by* (6.15). *Then, the energy*

$$e(t) := \tfrac{1}{2}[u(t), u(t)] + \tfrac{1}{2}(\dot{u}(t), \dot{u}(t))_H$$

is a constant function of t. Finally, problem (6.14) *has a unique solution.*

Proof. Conservation of energy follows from the following facts (we use the same notation of the proof of Proposition 6.3.1). First of all, we have the convergent V-orthogonal sum

$$u(t) = c(t) + s(t) = Pu_0 + t\, Pv_0 + \sum_{n=1}^{\infty}(c_n(t) + s_n(t)),$$

and the convergent H-orthogonal sum

$$\dot{u}(t) = Pv_0 + \sum_{n=1}^{\infty}(\dot{c}_n(t) + \dot{s}_n(t)).$$

Next, using that $[\phi_n, \phi_n] = \lambda_n^{-1} = \xi_n^2$, we prove that

$$\begin{aligned}
[c_n(t) + s_n(t), c_n(t) + s_n(t)] &= \xi_n^2 \cos^2(\xi_n t)|(u_0, \phi_n)_H|^2 + \sin^2(\xi_n t)|(v_0, \phi_n)_H|^2 \\
&\quad +2\cos(\xi_n t)\xi_n \sin(\xi_n t)\,(u_0, \phi_n)_H(v_0, \phi_n)_H,
\end{aligned}$$

while

$$\begin{aligned}
(\dot{c}_n(t) + \dot{s}_n(t), \dot{c}_n(t) + \dot{s}_n(t))_H &= \xi_n^2 \sin^2(\xi_n t)|(u_0, \phi_n)_H|^2 + \cos^2(\xi_n t)|(v_0, \phi_n)_H|^2 \\
&\quad -2\xi_n \sin(\xi_n t)\cos(\xi_n t)\,(u_0, \phi_n)_H(v_0, \phi_n)_H,
\end{aligned}$$

Hence

$$\begin{aligned}
2e(t) &= \sum_{n=1}^{\infty}[c_n(t) + s_n(t), c_n(t) + s_n(t)] + \|Pv_0\|_H^2 + \sum_{n=1}^{\infty}(\dot{c}_n(t) + \dot{s}_n(t), \dot{c}_n(t) + \dot{s}_n(t))_H \\
&= \|Pv_0\|_H^2 + \sum_{n=1}^{\infty}\xi_n^2|(u_0, \phi_n)_H|^2 + \sum_{n=1}^{\infty}|(v_0, \phi_n)_H|^2 \\
&= [u_0, u_0] + (v_0, v_0)_H.
\end{aligned}$$

The energy conservation property can also be proved directly, assuming that u has the regularity (6.16) and showing that

$$\dot{e}(t) = (\ddot{u}(t), \dot{u}(t))_H + [u(t), \dot{u}(t)] = (Au(t), \dot{u}(t))_H + [u(t), \dot{u}(t)] = 0$$

by the associated Green Identity. This also shows that if initial data are set to zero, then $\dot{u}(t) = 0$ for all t and therefore $u(t) \equiv u(0) = 0$. □

6.4 Strong solutions to non-homogeneous problems

The problem. The context of this section is exactly the one of the preceding sections (hypotheses are exposed in Section 6.1). In this case we want to study the problem

$$\left[\begin{array}{ll} \ddot{u}(t) = Au(t) + f(t) & t \in [0,\infty), \\ u(0) = 0, & \\ \dot{u}(0) = 0, & \end{array}\right. \tag{6.19}$$

where $f : [0,\infty) \to V_0$ is a continuous function. The solution of this problem can be expressed with a causal convolution, which is nothing but the representation using the formula of variation of constants (a.k.a. Duhamel's principle). This is where we will later make this theory meet the theory of retarded potentials and operators. The more general case, when $f(t) \in V$ will be treated in Section 6.6, by considering the solution corresponding to a right hand side $Pf(t)$.

The following lemma is traightforward to prove.

Lemma 6.4.1. *Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function, $\omega > 0$ and*

$$\alpha(t) := \omega^{-1} \int_0^t \sin(\omega(t-\tau))g(\tau)\mathrm{d}\tau.$$

Then $\alpha \in \mathcal{C}^2([0,\infty))$ and

$$\alpha(0) = \dot{\alpha}(0) = 0 \qquad \ddot{\alpha} + \omega^2\alpha = g \quad \textit{in } [0,\infty).$$

Proposition 6.4.2 (Duhamel's principle in series form)**.** *Let $f \in \mathcal{C}([0,\infty);V)$ and consider the functions*

$$u_n(t) := \Big(\int_0^t \xi_n^{-1} \sin(\xi_n(t-\tau))\,(f(\tau),\phi_n)_H \mathrm{d}\tau\Big)\phi_n \qquad n \geq 1. \tag{6.20}$$

Then

$$u(t) := \sum_{n=1}^\infty u_n(t)$$

is the unique solution of (6.19) *and*

$$u \in \mathcal{C}^2([0,\infty),H) \cap \mathcal{C}^1([0,\infty),V) \cap \mathcal{C}([0,\infty),D(A)).$$

Proof. Properties of the functions $u_n(t)$. Applying Lemma 6.4.1 to the functions $g := (f(\cdot),\phi_n)_H$, it follows that the functions u_n are in $\mathcal{C}^2([0,\infty),D(A))$. Also

$$\dot{u}_n(t) = \Big(\int_0^t \cos(\xi_n(t-\tau))\,(f(\tau),\phi_n)_H \mathrm{d}\tau\Big)\phi_n, \tag{6.21}$$

and

$$\begin{aligned} \ddot{u}_n(t) &= (f(t),\phi_n)_H\phi_n - \xi_n^2 u_n(t) && \text{(by Lemma 6.4.1)} \\ &= (f(t),\phi_n)_H\phi_n + Au_n(t) && (A\phi_n = -\xi_n^2\phi_n = -\lambda_n^{-1}\phi_n). \end{aligned} \tag{6.22}$$

It is also straightforward to see that, because of the orthogonality of the functions ϕ_n,

$$(u_n(t), u_m(\tau))_H = 0 \qquad n \neq m, \quad t, \tau \in [0, \infty), \tag{6.23a}$$

$$[u_n(t), u_m(\tau)] = 0 \qquad n \neq m, \quad t, \tau \in [0, \infty), \tag{6.23b}$$

$$(Au_n(t), Au_m(\tau))_H = 0 \qquad n \neq m, \quad t, \tau \in [0, \infty). \tag{6.23c}$$

Continuity in $D(A)$ and first initial condition. Using (6.13) (series expression for the norm in $D(A)$), we can write for all $t \leq T$

$$\begin{aligned}
\|u_n(t)\|^2_{D(A)} &= (\lambda_n^{-2} + \lambda_n^{-1}) \Big| \int_0^t \xi_n^{-1} \sin(\xi_n(t-\tau))(f(\tau), \phi_n)_H \mathrm{d}\tau \Big|^2 \\
&= (\lambda_n^{-1} + 1) \Big| \int_0^t \sin(\xi_n(t-\tau))(f(\tau), \phi_n)_H \mathrm{d}\tau \Big|^2 && (\xi_n^{-2}\lambda_n^{-1} = 1) \\
&\leq (\lambda_n^{-1} + 1) t \int_0^t |(f(\tau), \phi_n)_H|^2 \mathrm{d}\tau && \text{(Cauchy-Schwarz)} \\
&\leq T \int_0^T (\lambda_n^{-1} + 1) |(f(\tau), \phi_n)_H|^2 \mathrm{d}\tau =: M_n = M_n(T).
\end{aligned}$$

On the other hand, by the Monotone Convergence Theorem

$$\sum_{n=1}^{\infty} M_n = T \int_0^T \sum_{n=1}^{\infty} (\lambda_n^{-1} + 1) |(f(\tau), \phi_n)|^2 \mathrm{d}\tau \leq T \int_0^T \big(\|f(\tau)\|_V^2 + \|f(\tau)\|_H^2 \big) \mathrm{d}\tau,$$

where we have applied that $\{\phi_n\}$ is orthonormal in H and Proposition 6.2.2. Using Lemma 6.7.3 in the space $D(A)$ (note the orthogonality given by (6.23a) and (6.23c)), it follows that $u \in \mathcal{C}([0,T], D(A))$ for all $T > 0$ and thus $u \in \mathcal{C}([0,\infty), D(A))$. Since convergence of the series defining u is valid for all t (it is uniform in compact intervals), it also follows that

$$u(0) = \sum_{n=}^{\infty} u_n(0) = 0. \tag{6.24}$$

Also, using that $A : D(A) \to H$ is bounded, we can write

$$Au(t) = \sum_{n=1}^{\infty} Au_n(t) \qquad \text{in } H, \text{ uniformly in } [0,T] \text{ for all } T. \tag{6.25}$$

Differentiability in V and second initial condition. By Proposition 6.2.2 and the expression for $\dot{u}_n$ given in (6.21), we can write

$$\begin{aligned}
\|u_n(t)\|_V^2 + \|\dot{u}_n(t)\|_V^2 &= \lambda_n^{-1} \Big| \int_0^t \xi_n^{-1} \sin(\xi_n(t-\tau))(f(\tau), \phi_n)_H \mathrm{d}\tau \Big|^2 \\
&\quad + \lambda_n^{-1} \Big| \int_0^t \cos(\xi_n(t-\tau))(f(\tau), \phi_n)_H \mathrm{d}\tau \Big|^2 \\
&\leq (1 + \lambda_n^{-1}) t \int_0^t |(f(\tau), \phi_n)_H|^2 \mathrm{d}\tau \leq M_n \qquad \forall t \leq T.
\end{aligned}$$

We then use Lemma 6.7.4 in the space V to show that $u \in \mathcal{C}^1([0,T],V)$ for all T and thus $u \in \mathcal{C}^1([0,\infty),V)$. Since the series that defines u can be differentiated term by term (see Lemma 6.7.4) it also follows that

$$\dot{u}(0) = \sum_{n=1}^{\infty} \dot{u}_n(0) = 0. \tag{6.26}$$

Second order differentiability in H and differential equation. We first note that

$$\begin{aligned} \sum_{n=1}^{\infty} \ddot{u}_n(t) &= \sum_{n=1}^{\infty} \big(Au_n(t) + (f(t),\phi_n)_H\phi_n\big) && \text{(by (6.22)} \\ &= Au(t) + f(t), && \text{(by (6.25) and Lemma 6.7.5)} \end{aligned}$$

with convergence in H, uniformly for all $t \in [0,T]$ and for all $T > 0$. Since $\sum_{n=1}^{\infty} \dot{u}_n(t) = \dot{u}(t)$ in H, uniformly in $t \in [0,T]$, this implies that

$$\ddot{u} = Au + f \in \mathcal{C}([0,\infty),H). \tag{6.27}$$

Conclusion. Note that we have proved all desired continuity properties, the differential equation (6.27), and the initial conditions (6.24) and (6.26). Further, we have proved that the series $\sum_n u_n(t)$ converges in $D(A)$ uniformly on bounded intervals, that $\sum_n \dot{u}_n(t)$ converges in V uniformly on bounded intervals and $\sum_n \ddot{u}_n(t)$ converges in H uniformly on bounded intervals. $\square$

Proposition 6.4.3. *With the notation of the previous proposition, the following bounds hold for all $t \geq 0$:*

$$\begin{aligned} \|u(t)\|_{D(A)} &\leq \sqrt{1+C_\circ^2}\int_0^t \|f(\tau)\|_V \mathrm{d}\tau, && (6.28\text{a}) \\ \|Au(t)\|_H &\leq \int_0^t \|f(\tau)\|_V \mathrm{d}\tau, && (6.28\text{b}) \\ \|u(t)\|_V &\leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, && (6.28\text{c}) \\ \|\dot{u}(t)\|_V &\leq \int_0^t \|f(\tau)\|_V \mathrm{d}\tau, && (6.28\text{d}) \\ \|\dot{u}(t)\|_H &\leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau. && (6.28\text{e}) \end{aligned}$$

Proof. Let us fix a value of t and consider the functions

$$[0,t] \ni \tau \longmapsto g_n(\tau;t) := \xi_n^{-1}\sin(\xi_n(t-\tau))(f(\tau),\phi_n)_H\phi_n$$

as well as their sum

$$g(\tau;t) := \sum_{n=1}^{\infty} g_n(\tau;t), \tag{6.29}$$

assuming that this series converges. Since

$$\|g_n(\tau;t)\|^2_{D(A)} \le (\lambda_n^{-1}+1)|(f(\tau),\phi_n)_H|^2 \qquad \forall \tau\in[0,t] \quad \forall n \tag{6.30}$$

it follows that $g(\tau;t)$ is well defined for all τ and that

$$\|g(\tau;t)\|^2_{D(A)} = \sum_{n=1}^\infty \|g_n(\tau;t)\|^2_{D(A)} \le (1+C_\circ^2)\|f(\tau)\|^2_V \le C \quad \forall \tau\in[0,t].$$

This proves that $g(\,\cdot\,;t)\in L^2(0,t;D(A))\subset L^1(0,t;D(A))$ and therefore

$$\Big\|\int_0^t g(\tau;t)\mathrm{d}\tau\Big\|_{D(A)} \le \int_0^t \|g(\tau;t)\|_{D(A)}\mathrm{d}\tau \le \sqrt{1+C_\circ^2}\int_0^t \|f(\tau)\|_V\mathrm{d}\tau.$$

Since the functions $g_n(\,\cdot\,;t)$ are orthogonal in $L^2(0,t;D(A))$, the bound (6.30) also shows that the series (6.29) converges in $L^2(0,t;D(A))$ and therefore in $L^1(0,t;D(A))$. Thus

$$u(t)=\sum_{n=1}^\infty f_n(t)=\sum_{n=1}^\infty\int_0^t g_n(\tau;t)\mathrm{d}\tau=\int_0^t\sum_{n=1}^\infty g_n(\tau;t)\mathrm{d}\tau=\int_0^t g(\tau;t)\mathrm{d}\tau$$

and the proof of (6.28a) is finished. The other bounds are very similar and left to the reader as an exercise. □

6.5 Weak solutions of non-homogeneous equations

In this section we study problems of the form

$$\left[\begin{array}{ll} \ddot u(t)=Au(t)+f(t) & t\in[0,\infty),\\ u(0)=0, & \\ \dot u(0)=0, & \end{array}\right. \tag{6.31}$$

where $f:[0,\infty)\to H_0$ is a continuous function. We will look for some kind of weak solutions to this equation. First of all, note that if V' is the representation of the dual of V that follows from identifying H with H' (that is, we are considering the Courant triad $V\subset H\cong H'\subset V'$), then

$$\|v\|_{V'}\le C_\circ\|v\|_H \qquad \forall v\in H.$$

Moreover, A can be extended to a bounded operator $A:V\to V'$ by using the Green's Identity

$$\langle Au,v\rangle_{V'\times V} := -[u,v] \qquad \forall u,v\in V.$$

The series representation for A is still valid but convergence of the series

$$Au=-\sum_{n=1}^\infty \lambda_n^{-1}(u,\phi_n)_H\phi_n$$

is in V' (and not in H any more). A weak solution of (6.31) will thus be a solution of this problem with the differential equation valid in V' for all t, or equivalently, a solution of

$$\left[\begin{array}{ll} \langle \ddot{u}(t), v\rangle_{V'\times V} + [u(t), v] = (f(t), v)_H \quad \forall v \in V & t \in [0,\infty), \\ u(0) = 0, & \\ \dot{u}(0) = 0, & \end{array}\right. \tag{6.32}$$

Note that the solution given by Proposition 6.4.2 is easily seen to be a weak solution.

Proposition 6.5.1. *For any continuous $f : [0,\infty) \to H_0$, problem (6.32) has a unique solution*

$$u \in \mathcal{C}^2([0,\infty); V') \cap \mathcal{C}^1([0,\infty); H) \cap \mathcal{C}^0([0,\infty); V) \tag{6.33}$$

and the following bounds hold for all $t \geq 0$:

$$\begin{aligned} C_\circ^{-1}\|u(t)\|_H \leq \|u(t)\|_V &\leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \\ \|\dot{u}(t)\|_H &\leq \int_0^t \|f(\tau)\|_H \mathrm{d}\tau. \end{aligned}$$

Moreover, the function

$$w(t) := \int_0^t u(\tau)\mathrm{d}\tau \tag{6.34}$$

is in $\mathcal{C}([0,\infty); D(A))$.

Proof. We first consider the following modified problem

$$\left[\begin{array}{ll} \ddot{v}(t) = Av(t) + G^{1/2}f(t) & t \in [0,\infty), \\ v(0) = 0, & \\ \dot{v}(0) = 0. & \end{array}\right. \tag{6.35}$$

Since $G^{1/2}f : [0,\infty) \to V_0$ is continuous, we can apply Propositions 6.4.2 and 6.4.3 and we obtain that problem (6.19) has a solution

$$v \in \mathcal{C}^2([0,\infty); H) \cap \mathcal{C}^1([0,\infty); V) \cap \mathcal{C}^0([0,\infty); D(A)). \tag{6.36}$$

From Proposition 6.2.2 it follows that

$$\|Av(t)\|_H \leq \int_0^t \|G^{1/2}f(\tau)\|_V \mathrm{d}\tau = \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \tag{6.37}$$

$$\|v(t)\|_V \leq \int_0^t \|G^{1/2}f(t)\|_H \mathrm{d}\tau \leq C_\circ \int_0^t \|f(\tau)\|_H \mathrm{d}\tau, \tag{6.38}$$

$$\|\dot{v}(t)\|_V \leq \int_0^t \|G^{1/2}f(\tau)\|_V \mathrm{d}\tau = \int_0^t \|f(\tau)\|_H \mathrm{d}\tau. \tag{6.39}$$

Consider now the operator

$$G^{-1/2}w := \sum_{n=1}^\infty \lambda_n^{-1/2}(w, \phi_n)_H \phi_n = (-A)^{1/2}w.$$

(The notation is slightly misleading, because $G^{1/2}$ is invertible only when we eliminate rigid motions from the problem.) It is easy to prove that $G^{-1/2}$ is bounded $D(A) \to V$, $V \to H$ and therefore $H \to V'$. Note also that as operators $D(A) \to V'$

$$G^{-1/2}A = AG^{-1/2}, \tag{6.40}$$

which can be proved by a density argument proving the result for the eigenfunctions ϕ_n.

We then define $u(t) := G^{-1/2}v(t)$ and note that the regularity (6.36) for $v(t)$ implies the regularity (6.33) for $u(t)$. Initial conditions for u are satisfied for obvious reasons. Finally, with equalities in V'

$$\begin{aligned} \ddot u(t) &= G^{-1/2}\ddot v(t) && (G^{-1/2} \text{ is bounded from } H \text{ to } V') \\ &= G^{-1/2}Av(t) + G^{-1/2}G^{1/2}f(t) && (v \text{ satisfies (6.35)}) \\ &= AG^{-1/2}v(t) + f(t) && ((6.40) \text{ and } f \text{ takes values in } H_0) \\ &= Au(t) + f(t). \end{aligned}$$

In this sense, $u(t)$ is a solution of (6.31) by understading the equation to be held in the space V'. Note also that $u(t) \in V_0$ for all t, which implies that $v(t) = G^{1/2}u(t)$, and therefore

$$\|Av(t)\|_H = \|AG^{1/2}u(t)\|_H = \|u(t)\|_V, \qquad \|v(t)\|_V = \|G^{1/2}u(t)\|_V = \|u(t)\|_H,$$

and

$$\|\dot v(t)\|_V = \|G^{1/2}\dot u(t)\|_V = \|\dot u(t)\|_H.$$

These equalities and (6.37)-(6.39) finish the bounds for u.

Let us finally consider the function w in (6.34). It is clear that $w \in \mathcal{C}^2([0,\infty)'H) \cap \mathcal{C}^1([0,\infty);V)$ and that

$$Aw(t) = \ddot w(t) - \int_0^t f(\tau)\mathrm{d}\tau \qquad (\text{in } V') \qquad \forall t \geq 0. \tag{6.41}$$

However, the right hand side of (6.38) is a continuous function with values in H and therefore, so is Aw. □

Proposition 6.5.2. *Let $f \in \mathcal{C}^1([0,\infty);H_0)$ with $f(0) = 0$. Then the problem*

$$\left[\begin{array}{ll} \ddot u(t) = Au(t) + f(t) & t \in [0,\infty), \\ u(0) = 0, & \\ \dot u(0) = 0, & \end{array}\right.$$

has a unique solution

$$u \in \mathcal{C}^2([0,\infty);H) \cap \mathcal{C}^1([0,\infty);V) \cap \mathcal{C}([0,\infty);D(A)) \tag{6.42}$$

satisfying the bounds of Proposition 6.5.1 as well as for all $t \geq 0$

$$\|Au(t)\|_H \;\leq\; \|f(t)\|_H + \int_0^t \|\dot f(\tau)\|_H \mathrm{d}\tau \leq 2\int_0^t \|\dot f(\tau)\|_H\mathrm{d}\tau, \tag{6.43a}$$

$$\|\dot u(t)\|_V \;\leq\; \int_0^t \|\dot f(\tau)\|_H\mathrm{d}\tau. \tag{6.43b}$$

Proof. Let first $z : [0, \infty) \to V$ be the solution to

$$\left[\begin{array}{l} \langle \ddot{z}(t), v\rangle_{V'\times V} + [z(t), v] = (\dot{f}(t), v)_H \quad \forall v \in V \qquad t \in [0, \infty), \\ z(0) = 0, \\ \dot{z}(0) = 0, \end{array}\right.$$

and define

$$u(t) := \int_0^t z(\tau)\mathrm{d}\tau.$$

By Proposition 6.5.1, it follows that u satisfies (6.42). (Note that we are integrating once in the time variable, which increases time regularity by one index.) Integrating the weak differential equation satisfied by z and using that $f(0) = 0$, it follows that

$$(\ddot{u}(t), v)_H + [u(t), v] = \langle \ddot{u}(t), v\rangle_{V'\times V} + [u(t), v] = (f(t), v)_H \qquad \forall v \in V.$$

Therefore,

$$\ddot{u}(t) - f(t) = Au(t) \qquad \text{in } V' \quad t \geq 0,$$

but since both sides are in H, this is just the strong differential equation. Finally

$$\|Au(t)\|_H \leq \|\ddot{u}(t)\|_H + \|f(t)\|_H = \|\dot{z}(t)\|_H + \|f(t)\|_H$$

and (see Proposition 6.5.1)

$$\|z(t)\|_V \leq \int_0^t \|\dot{f}(\tau)\|_H \mathrm{d}\tau, \qquad \|\dot{z}(t)\|_H \leq \int_0^t \|\dot{f}(\tau)\|_H \mathrm{d}\tau,$$

which proves (6.43). □

6.6 Purely kinetic motion

For some of the arguments below it will be convenient to make use of an orthonormal basis of the finite dimensional space of rigid motions, M, so that

$$Pu = \sum_{n=1}^K (u, m_n)_H m_n \qquad \forall u \in H.$$

Proposition 6.6.1. *For any continuous $f : [0, \infty) \to H$, the function*

$$m_f(t) := \int_0^t (t-\tau) P f(\tau)\mathrm{d}\tau = \sum_{n=1}^K \Big(\int_0^t (t-\tau)(f(\tau), m_n)_H \mathrm{d}\tau \Big) m_n, \tag{6.44}$$

is the unique solution of

$$\left[\begin{array}{l} \ddot{u}(t) = Au(t) + Pf(t) \qquad t \in [0, \infty), \\ u(0) = 0, \\ \dot{u}(0) = 0, \end{array}\right.$$

Moreover $m_f \in \mathcal{C}^2([0, \infty); M)$ (with any norm in the finite dimensional space M).

Proof. Note that

$$\dot{m}_f(t) = \int_0^t Pf(\tau)\mathrm{d}\tau \qquad \ddot{m}_f(t) = Pf(t) = Am_f(t) + Pf(t).$$

Initial conditions are straightforward to verify as well. Note also that since

$$\langle \ddot{m}_f(t), v\rangle_{V'\times V} + [m_f(t), v] = \langle \ddot{m}_f(t), v\rangle_{V'\times V} = (\ddot{m}_f(t), v)_H = (Pf(t), v)_H \qquad \forall v \in V,$$

then m_f is also a weak solution of the equations. □

Final remark. For problems

$$\left[\begin{array}{ll} \ddot{u}(t) = Au(t) + f(t) & t \in [0,\infty), \\ u(0) = 0, & \\ \dot{u}(0) = 0, & \end{array}\right.$$

with $f : [0,\infty) \to V$ or

$$\left[\begin{array}{ll} \langle \ddot{u}(t), v\rangle_{V'\times V} + [u(t), v] = (f(t), v)_H \quad \forall v \in V & t \in [0,\infty), \\ u(0) = 0, & \\ \dot{u}(0) = 0, & \end{array}\right.$$

with $f : [0,\infty) \to H$, we can decompose the solution as the sum of the rigid motion (6.44), which satisfies

$$\|m_f(t)\|_V = \|m_f(t)\|_H \qquad \text{and} \qquad \|Am_f(t)\|_H = 0$$

plus the solution of the problem with $f(t) - Pf(t)$ as right hand side. For the bounds of Propositions 6.4.3 and 6.5.1, note that

$$\int_0^t \|f(\tau) - Pf(\tau)\|_H \mathrm{d}\tau \le \int_0^t \|f(\tau)\|_H \mathrm{d}\tau$$

and

$$\int_0^t \|f(\tau) - Pf(\tau)\|_V \mathrm{d}\tau = \int_0^t |f(\tau)|_V \mathrm{d}\tau, \qquad |f(\tau)|_V^2 = [f(\tau), f(\tau)].$$

Also $|u(t)|_V = |u(t) - m_f(t)|_V = \|u(t) - m_f(t)\|_V$.

6.7 Background material

For ease of reference, we are including here some basic results on functional analysis, a small block thereof related to spectral decompositions of compact operators and a second group related to uniform convergence of series. Proofs of these results can be found in any basic text on functional analysis.

Hilbert-Schmidt theory

We start the section with a slightly restricted version of the Hilbert-Schmidt Theorem for compact selfadjoint operators in Hilbert space.

Theorem 6.7.1 (Hilbert-Schmidt Decomposition)**.** *Let H_0 be an infinite dimensional real Hilbert space. Let $G : H_0 \to H_0$ be a compact (therefore bounded) self-adjoint and positive definite (therefore injective) linear operator. Then there exist a non-increasing sequence of positive numbers $\lim_{n\to\infty} \lambda_n = 0$ and a complete orthogonal set $\{\phi_n\} \subset H_0$ such that*

$$G = \sum_{n=1}^{\infty} \lambda_n(\,\cdot\,, \phi_n)\,\phi_n, \tag{6.45}$$

with uniform convergence of the series.

Simple consequences plus some remarks.

(a) Uniform convergence of the series refers to convergence in operator norm. Therefore

$$\lim_{N\to\infty} \|G - \sum_{n=1}^{N} \lambda_n(\,\cdot\,, \phi_n)\phi_n\| = 0.$$

(b) It is obvious that

$$G\phi_n = \lambda_n \phi_n$$

and therefore the decomposition (6.45) singles out eigenvalues and eigenvectors of G.

(c) Because $\{\phi_n\}$ is a complete orthonormal set in H_0 we can write

$$u = \sum_{n=1}^{\infty} (u, \phi_n)\phi_n \qquad \forall u \in H_0. \tag{6.46}$$

In particular, the hypotheses of the Hilbert-Schmidt theorem, as given here, cannot hold unless H_0 is a separable Hilbert space. A more general result can be easily shown by assuming that G is only positive semidefinite. In this case, the decomposition (6.45) still holds, but (6.46) only holds in the orthogonal complement of the kernel of G, a subspace of H_0 that can be non-separable.

(d) It can be easily proved that the decomposition (6.45) shows all possible eigenvalues of G and that eigenfunctions are necessarily linear combinations of those given in the Hilbert basis.

(e) The reciprocal statement to that of Proposition 6.7.1 is easy to prove: if there exist sequences $\{\lambda_n\}$ and $\{\phi_n\}$ in the conditions above ($\lambda_n > 0$, $\lim_{n\to\infty} \lambda = 0$, $\{\phi_n\}$ complete orthonormal in H_0), then the operator G is a compact selfadjoint positive definite linear operator in H_0.

Picard's criterion goes one step further to characterize the range of an operator of the form (6.45) with the conditions above for the eigenvalues and eigenvectors.

Theorem 6.7.2 (Picard's Criterion)**.** *Let $G : H_0 \to H_0$ be an operator in the conditions of Proposition 6.7.1 and consider its Hilbert-Schmidt decomposition* (6.45)*. Then $f \in R(G)$ if and only if*

$$\sum_{n=1}^{\infty} \lambda_n^{-2} |(f, \phi_n)|^2 < \infty. \tag{6.47}$$

More remarks and observations.

(a) Picard's criterion is actually proved by finding the inverse, which is actually a rather simple thing to do:

$$R(G) \ni f \qquad \longmapsto \qquad u := \sum_{n=1}^{\infty} \lambda_n^{-1} (f, \phi_n) \phi_n.$$

It is clear that this is an element of H_0 and that $Gu = f$.

(b) The condition (6.47) can actually be used to endow $R(G)$ with a norm:

$$\|f\|_{R(G)}^2 := \sum_{n=1}^{\infty} \lambda_n^{-2} |(f, \phi_n)|^2$$

This norm makes $R(G)$ a Hilbert that is compact and densely embedded in H. Furthermore, $R : H \to R(G)$ is an isometric isomorphism.

Some continuity tests

Lemma 6.7.3. *Let X be a Hilbert space and I a closed interval in $\mathbb{R}$. Assume that $c_n : I \to X$ are continuous functions such that*

$$(c_n(t), c_m(t))_X = 0 \quad \forall n \neq m \qquad \forall t \in I$$

and

$$\|c_n(t)\|_X^2 \leq M_n \quad \forall t \in I \qquad \text{where} \qquad \sum_{n=1}^{\infty} M_n < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} c_n(t)$$

converges uniformly to a function in $\mathcal{C}(I; X)$.

Proof. Consider the partial sums

$$s_N := \sum_{n=1}^{N} c_n \in \mathcal{C}(I; X)$$

and note that for $M > N$

$$\|s_M(t) - s_N(t)\|_X^2 = \Big\| \sum_{n=N+1}^{M} c_n(t) \Big\|_X^2 = \sum_{n=N+1}^{M} \|c_n(t)\|_X^2 < \sum_{n=N+1}^{M} M_n \quad \forall t \in I.$$

This proves that s_N is Cauchy in $\mathcal{C}(I;X)$ and therefore, it is convergent in this space. □

A second variant of this result includes term by term differentiation. The proof is similar.

Lemma 6.7.4. *Let X be a Hilbert space and I a closed interval in $\mathbb{R}$. Assume that $c_n \in \mathcal{C}^1(I;X)$ satisfy*

$$(c_n(t), c_m(\tau))_X = 0 \quad \forall n \neq m \qquad \forall t, \tau \in I,$$

and

$$\|c_n(t)\|_X^2 + \|\dot{c}_n(t)\|_X^2 \leq M_n \quad \forall t \in I \qquad \text{where} \qquad \sum_{n=1}^{\infty} M_n < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} c_n(t) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \dot{c}_n(t)$$

converge in $\mathcal{C}(I;X)$ and the derivative of the first function is the second one.

Proof. The hypothesis of orthogonality for all values of t, τ implies that

$$(\dot{c}_n(t), \dot{c}_m(t))_X = 0 \qquad \forall n \neq m \qquad \forall t \in I.$$

Using the notation of the proof of Lemma 6.7.3, we thus prove that for $M > N$

$$\|s_M(t) - s_N(t)\|_X^2 + \|\dot{s}_M(t) - \dot{s}_N(t)\|_X^2 \leq \sum_{n=N+1}^{M} M_n \qquad \forall t \in I,$$

from where the result follows. □

Lemma 6.7.5. *Let $f : I \to X$ be continuous and I is a closed bounded interval. Assume that X has a Hilbert basis $\{\phi_n\}$. Then, the series*

$$f(t) = \sum_{n=1}^{\infty} (f(t), \phi_n)_X \phi_n \tag{6.48}$$

converges in $\mathcal{C}(I;X)$.

Proof. It is clear that the series (6.48) converges in X for all t and in particular

$$a_N(t) := \Big\| \sum_{n=1}^{N} (f(t), \phi_n)_X \phi_n \Big\|_X^2 = \sum_{n=1}^{N} |(f(t), \phi_n)_X|^2 \overset{N\to\infty}{\longrightarrow} \sum_{n=1}^{\infty} |(f(t), \phi_n)_X|^2 = \|f(t)\|_X^2$$

for all t. The functions $a_N : I \to \mathbb{R}$ are continuous and increase monotonically to its continuous limit $a(t) := \|f(t)\|_X^2$. By Dini's Theorem, $a_N \to a$ uniformly. Finally

$$\Big\| f(t) - \sum_{n=1}^{\infty} (f(t), \phi_n)_X \phi_n \Big\|_X^2 = a(t) - a_N(t) \qquad \forall t \in I, \quad \forall N,$$

which proves the result. □

6.8 Exercises

1. (Section 6.2) Consider the spaces

$$H := L^2(0,L), \quad V := H^1_0(0,L), \quad D(A) := H^1_0(0,L) \cap H^2(0,L).$$

Describe the spaces (and their associated norms) using Fourier sines series. Note that the Fourier sine series is the series of eigenfunctions for the one dimensional problem

$$-u'' = f \quad \text{in } (0,L), \qquad u(0) = u(L) = 0$$

Follow carefully where the constant of Poincaré's inequality appears.

2. (Section 6.2) Describe the spaces (and their norms)

$$H := L^2(0,L), \quad V := H^1(0,L), \quad D(A) := \{u \in H^2(0,L) \,:\, u'(0) = u'(L) = 0\}$$

using Fourier cosine series. Note that the Fourier cosine series is the series of eigenfunctions for the one dimensional problem

$$-u'' = f \quad \text{in } (0,L), \qquad u'(0) = u'(L) = 0$$

and that there is a one-dimensional space of rigid motions.

3. (Section 6.2) Study the Green's operator associated to the problem with periodic boundary conditions

$$-u'' = f \quad \text{in } (0,L), \qquad u(0) = u(L), \qquad u'(0) = u'(L).$$

4. (Section 6.2) **Associated Hilbert scales.** In the notation of Section 6.2, for $s \geq 0$ we consider the space

$$H_s := \{u \in H_0 \,:\, \|u\|_s < \infty\}, \quad \text{where} \quad \|u\|_s^2 := \sum_{n=1}^{\infty} \lambda_n^{-s} |(u, \phi_n)_H|^2.$$

Note that $H_1 = V_0 = R(G^{1/2})$ and $H_2 = D(A) \cap H_0 = R(G)$. Note also that by Picard's Criterion $H_s = R(G^{s/2})$, where

$$G^{s/2} u := \sum_{n=1}^{\infty} \lambda_n^{-s/2} (u, \phi_n)_H \phi_n.$$

(a) Show that H_s si a Hilbert space. (**Hint.** You can easily show H_s to be isomorphic to a space of sequences with a weighted ℓ^2 norm.)

(b) Find the best estimate for $C(r,s) > 0$ such that

$$\|u\|_s \leq C(r,s) \|u\|_r \qquad \forall u \in H_r \qquad r \geq s.$$

(c) Show that if $r > s$, then the inclusion $H_r \subset H_s$ is dense and compact.

(d) Show that $G : H_s \to H_{s+2}$ and $A : H_{s+2} \to H_s$ are bounded for all $s \geq 0$.

5. (Section 6.3) **The evolution group.** Using notation of Section 6.3, consider the matrix of operators

$$\mathbb{S}(t) := \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix},$$

where

$$\begin{aligned}
S_{11}(t)u &:= \ Pu + \sum_{n=1}^{\infty} \cos(\xi_n t)(u, \phi_n)_H \phi_n, \\
S_{12}(t)v &:= t\, Pv + \sum_{n=1}^{\infty} \xi_n^{-1} \sin(\xi_n t)(v, \phi_n)_H \phi_n, \\
S_{21}(t)u &:= \qquad - \sum_{n=1}^{\infty} \xi_n \sin(\xi_n t)(u, \phi_n)_H \phi_n, \\
S_{22}(t)v &:= \ Pv + \sum_{n=1}^{\infty} \cos(\xi_n t)(v, \phi_n)_H \phi_n.
\end{aligned}$$

These operators are defined for all $t \in \mathbb{R}$. Note that, at least formally, the second row of $\mathbb{S}(t)$ is the derivative of the first one.

(a) Show that $\mathbb{S}(t)$ defines a bounded operator $V \times H \to V \times H$.

(b) Show that $\mathbb{S}(t)$ is an isometry in $V \times H$ for all $t \in \mathbb{R}$ and that $\mathbb{S}(0)$ is the identity operator.

(c) Show that $\mathbb{S}(t)\mathbb{S}(\tau) = \mathbb{S}(t + \tau)$ for all $t, \tau \in \mathbb{R}$. (**Hint.** This can be proved on a dense subspace.)

(d) Show that $\mathbb{S}(-t) = \mathbb{S}(t)^{-1}$. (**Hint.** This is an easy consequence of the above.)

(e) Show that $\mathbb{S}(t)$ is also bounded $D(A) \times V \to D(A) \times V$.

6. (Section 6.4) Prove all the bounds of Proposition 6.4.3.

Chapter 7

Time domain analysis of the single layer potential

In this chapter we are going to develop a systematic approach to a purely time domain analysis of the single layer potential and operator, the Galerkin semidiscrete error operators and the Galerkin solvers, namely, we want to prove estimates of

$$\mathcal{S} * \lambda, \qquad \mathcal{V} * \lambda, \qquad \mathcal{K}^t * \lambda,$$

of its semidiscrete inverses

$$\mathcal{G}^h_\lambda * \beta, \qquad \mathcal{G}^h_u * \beta = \mathcal{S} * \mathcal{G}^h_\lambda * \beta$$

and of the error operators

$$\mathcal{E}^h_\lambda * \lambda = \mathcal{G}^h_\lambda * \mathcal{V} * \lambda - \lambda, \qquad \mathcal{E}^h_u * \lambda = \mathcal{S} * \mathcal{E}^h_\lambda * \lambda.$$

The analysis of the semidiscrete operators will be developed in the usual 'for all h' spirit of Chapter 5, meaning that we will look for bounds independent of the discrete space $X_h \subset H^{-1/2}(\Gamma)$. We will only use that the space X_h is closed. Given the fact that when $X_h = H^{-1/2}(\Gamma)$, $\mathcal{G}^h_\lambda = \mathcal{V}^{-1}$, as a byproduct we will obtain an analysis of the operators $\mathcal{V}^{-1} * \beta$ and $\mathcal{S} * \mathcal{V}^{-1} * \beta$.

7.1 The cut-off process

Much of the technical effort of this chapter is related on the way of fitting transmission problems for the wave equation in the frame of the second order problems of Chapter 6. We are going to use finite speed of propagation of waves generated by potentials to cut-off the domain at a given distance from the scatterer. In this way, the potentials will coincide with the solution of wave propagation problems in bounded domains in a time interval $[0, T]$. Taking larger values of T, and following carefully all constants depending on the cut-off domain, we will be able to provide a full analysis for all times.

The cut-off geometry. Let $R > 0$ be such that

$$\overline{\Omega^-} \subset B(\mathbf{0}; R) = \{\mathbf{y} \in \mathbb{R}^d \,:\, |\mathbf{y}| < R\} =: B_0$$

and let

$$B_T := B(\mathbf{0}; R + T).$$

Consider also the distance of the boundary of B_0 to Γ:

$$\delta := \mathrm{dist}(\partial B_0, \Gamma) := \min\{|\mathbf{x} - \mathbf{y}| \,:\, \mathbf{x} \in \Gamma, |\mathbf{y}| = R\} > 0.$$

The trace operator and normal derivative operators on ∂B_T will have their own notation, making T explicit

$$\begin{aligned} \gamma_T &: H^1(B_T \setminus \overline{\Omega^-}) \to H^{1/2}(\partial B_T), \\ \partial_T^\nu &: H^1_\Delta(B_T \setminus \overline{\Omega^-}) \to H^{-1/2}(\partial B_T). \end{aligned}$$

Proposition 7.1.1 (Finite speed of propagation of potentials)**.** *Let* $\lambda \in \mathrm{TD}(H^{-1/2}(\Gamma))$, $\varphi \in \mathrm{TD}(H^{1/2}(\Gamma)$ *and* $u := \mathcal{S} * \lambda + \mathcal{D} * \varphi$*. Then*

$$\mathrm{supp}\, \gamma_T u \subset [T + \delta, \infty) \qquad \mathrm{supp}\, \partial_T^\nu u \subset [T + \delta, \infty).$$

Proof. Note that $\gamma_T u \in \mathrm{TD}(H^{1/2}(\partial B_T))$ and $\partial_T^\nu u \in \mathrm{TD}(H^{-1/2}(\partial B_T))$. The result follows from the causality analysis of Section 3.6, in particular from Proposition 3.6.2. □

Some inequalities. It will be convenient to keep track of several constants related to classical inequalities.

(a) The Poincaré-Friedrichs inequality in B_T:

$$\|v\|_{B_T} \le C_T \|\nabla v\|_{B_T} \qquad \forall v \in H^1_0(B_T). \tag{7.1}$$

Note that we can take

$$C_T = C_0(1 + T/R). \tag{7.2}$$

(b) Boundedness of the trace operators:

$$\|\gamma^\pm u\|_{1/2,\Gamma} \le C_\gamma \|u\|_{1,B_0\setminus\Gamma} \qquad \forall u \in H^1(B_0 \setminus \Gamma).$$

(c) Boundedness of the normal derivative operator:

$$\|\partial_\nu^\pm u\|_{-1/2,\Gamma} \le C_\nu \Big(\|\nabla u\|^2_{B_0\cap\Omega^\pm} + \|\Delta u\|^2_{B_0\cap\Omega^\pm} \Big)^{1/2} \qquad \forall u \in H^1_\Delta(B_0 \setminus \Gamma). \tag{7.3}$$

(d) A fixed two sided lifting of the trace. We consider an operator $\gamma^\dagger : H^{1/2}(\Gamma) \to H^1_0(B_0)$ and a constant $C^\dagger_\gamma > 0$ such that

$$\gamma\gamma^\dagger\beta = \beta, \qquad \|\gamma^\dagger\beta\|_{1,B_0} \le C^\dagger_\gamma \|\beta\|_{1/2,\Gamma} \qquad \forall \beta \in H^{1/2}(\Gamma). \tag{7.4}$$

The space

$$\mathbb{X}_T := H^1_0(B_T) \cap H^1_\Delta(B_T \setminus \Gamma),$$

endowed with the norm

$$\|u\|^2_{\mathbb{X}_T} := \|\nabla u\|^2_{B_T} + \|\Delta u\|_{B_T\setminus\Gamma},$$

will play a prominent role in all what follows.

The general plan. Following a several step process which mimicks in the time domain what we did in the Laplace domain (Sections 5.3 and 5.4), we will work in a systematic way.

- Recognize the transmission problem for the wave equation that we want to analyze. Cut then off the domain with the boundary ∂B_T and impose a Dirichlet boundary condition.
- Identify the underlying dynamical system frame $D(A) \subset V \subset H$ and check all needed properties to fit in the frame of Chapter 6 (see below). Transmission conditions that appear in the definition of V will be called *essential*, while those that are imposed in the definition of $D(A)$ will be called *natural*.
- Find a lifting process related to the non-homogeneous transmission conditions.
- Prove existence of strong solutions to the wave propagation problem in bounded domain.
  - When transmission conditions are natural this will be doable in a single step (using strong solutions of non-homogeneous second order equations after lifting the boundary condition).
  - When transmission conditions are essential, the lifting will lead to a weak solutions of a non-homogeneous second order equation, and a correction step (using higher data regularity) will then be applied to obtain bounds for all desired quantities.
- Show finite speed of propagation for the strong solution of the wave propagation problem in bounded domain. This will be phrased as a waiting time property.
- Identify potentials and strong solutions in the interval $[0, T]$ and use the previous bounds to estimate the potentials at time $t = T$.
- Finally use density arguments and the convolutional character of potentials to refine the continuity requirements of data. In the three cases studied in this section, this part will be collected in the final section.

The dynamical system framework (a checklist). In all examples below, we will need to identify the spaces

$$D(A) \subset V \subset H,$$

and their norms and inner products. (There will not be rigid motions here.) Recall that, in absence of rigid mitions, we denote by $[\,\cdot\,,\,\cdot\,]$ the inner product of V. We also have to define the operator A, which will be implicit to the definition of $D(A)$. The checklist consists then of the following points:

- Verify that the injection $V \subset H$ is compact and dense.

- Find the constant related to the (generalized Poincaré) inequality

$$\|u\|_H \le C_\circ \|u\|_V \qquad \forall u \in V.$$

- Verify the associated Green identity:

$$[u, v] + (Au, v)_H = 0 \qquad \forall u \in D(A), v \in V.$$

- Verify the surjectivity of $I - A$, i.e., given $f \in H$ find

$$u \in D(A) \qquad \text{such that} \qquad -Au + u = f.$$

A word on causal functions and $[0, \infty)$. The dynamical system approach will work with data $[0, \infty) \to X$. In particular we will use the spaces

$$\mathcal{C}_0^k([0, \infty); X) := \{u \in \mathcal{C}^k([0, \infty); X) \,:\, u^{(\ell)}(0) = 0 \quad \ell \le k - 1\}.$$

Tempered or polynomial behavior at infinity will be often assumed to simplify arguments (mainly to have Laplace transforms or to be in the time-domain class $\mathrm{TD}(X)$.) However, given the fact that all operators involved in this analysis are causal convolution operators, the behavior of data functions at infinity will not be relevant. Unlike what we did in Chapter 2 (when we were still trying to distinguish derivatives), the dot symbol will be used for strong classical differentiation of functions $[0, \infty) \to X$ and for distributional differentiation of X-valued causal distributions.

Extension by zero to negative values of the time variable will be done using the following notation

$$u \in \mathcal{C}([0, \infty); X) \quad \longmapsto \quad Eu(t) := \begin{cases} u(t), & t \ge 0, \\ 0, & t < 0. \end{cases}$$

For ease of reference we now gather some simple properties of the extension operator E.

Lemma 7.1.2. *Let $u \in \mathcal{C}([0, \infty); X)$.*

(a) *Eu is a causal X-valued distribution.*

(b) *If $\|u(t)\| \le Ct^m$ for all $t \ge 1$ with some positive m, then $Eu \in \mathrm{TD}(X)$.*

(c) *If $A \in \mathcal{B}(X, Y)$, then*

$$AEu = EAu,$$

where EAu is the extension of the continuous function $Au : [0, \infty) \to Y$ to an Y-valued distribution.

(d) *If $u \in \mathcal{C}^1([0, \infty); X)$, then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(Eu) = E\dot{u} + \delta_0 \otimes u(0).$$

In particular, if $u \in \mathcal{C}_0^1([0, \infty); X)$, then

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(Eu) = E\dot{u}.$$

7.2 Potentials and operators

In order to study $u = \mathcal{S} * \lambda$, its trace $\mathcal{V} * \lambda = \gamma(\mathcal{S} * \lambda)$ and the average of its normal derivatives $\mathcal{K}^t * \lambda = \{\!\{\partial_\nu(\mathcal{S} * \lambda)\}\!\}$, we consider the following cut-off problem: we look for $u_T : [0,\infty) \to H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)$ satisfying

$$\ddot{u}_T(t) = \Delta u_T(t) \qquad \forall t \geq 0, \tag{7.5a}$$

$$\gamma_T u_T(t) = 0 \qquad \forall t \geq 0, \tag{7.5b}$$

$$[\![\partial_\nu u_T]\!](t) = \lambda(t) \qquad \forall t \geq 0, \tag{7.5c}$$

$$u_T(0) = \dot{u}_T(0) = 0. \tag{7.5d}$$

The Laplace operator in (7.5a) is in the sense of distributions in $B_T \setminus \Gamma$ and this equality holds in the space $L^2(B_T) \equiv L^2(B_T \setminus \Gamma)$. The equation (7.5b) is imposed in $H^{1/2}(\partial B_T)$ and will be substituted by the implicit demand that

$$u_T : [0,\infty) \to \mathbb{X}_T = H^1_0(B_T) \cap H^1_\Delta(B_T \setminus \Gamma).$$

The jump condition (7.5c) will be imposed in $H^{-1/2}(\Gamma)$.

The dynamical system framework. Consider the spaces

$$\begin{aligned} H &:= L^2(B_T), \\ V &:= H^1_0(B_T), \\ D(A) &:= H^1_0(B_T) \cap H^2(B_T) = H^1_0(B_T) \cap H^1_\Delta(B_T) \\ &= \{u \in V \,:\, \Delta u \in L^2(B_T \setminus \Gamma), \quad [\![\partial_\nu u]\!] = 0\}, \end{aligned}$$

with

$$\|u\|_H := \|u\|_{B_T}, \qquad [u,v] := (\nabla u, \nabla v)_{B_T}$$

and where A is the distributional Laplacian in $B_T \setminus \Gamma$. Note that since functions in $D(A)$ do not have jumps for the trace and normal derivative across Γ, we can equally take A to be the Laplace operator in B_T. We now proceed with the checklist:

- Since $\mathcal{D}(B_T) \subset V$, the density of V in H follows. Compactness of the inclusion follows from Rellich's Compactness Theorem.
- The generalized Poincaré inequality is just the Poincaré inequality (7.1) and therefore $C_\circ = C_T$.
- The associated Green identity is just Green's Identity

  $$(\nabla u, \nabla v)_{B_T} + (\Delta u, v)_{B_T} = 0 \qquad \forall u \in D(A) \subset H^1_\Delta(B_T) \qquad \forall v \in V = H^1_0(B_T).$$

- Given $f \in H = L^2(B_T)$ we can easily find $u \in H^1(B_T)$ satisfying

  $$-\Delta u + u = f \quad \text{in } B_T \qquad \gamma_T u = 0.$$

  Therefore $u \in D(A)$ and $-Au + u = f$.

The lifting operator. Consider the operator $L : H^{-1/2}(\Gamma) \to V = H_0^1(B_T)$ given by the solution of the variational problem

$$\left[\begin{array}{l} u_0 \in H_0^1(B_T), \\ (\nabla u_0, \nabla v)_{B_T} + (u_0, v)_{B_T} = \langle \lambda, \gamma v\rangle_\Gamma \qquad \forall v \in H_0^1(B_T), \end{array}\right.$$

or equivalently, the transmission problem

$$\begin{aligned} u_0 &\in H_0^1(B_T), \\ -\Delta u_0 + u_0 &= 0 \qquad \text{in } B_T \setminus \Gamma, \\ [\![\partial_\nu u_0]\!] &= \lambda. \end{aligned}$$

(Note that the boundary condition $\gamma_T u_0 = 0$ is implicit in the fact that $u_0 \in H_0^1(B_T)$.) Since we can easily bound

$$\|u_0\|_{\mathbb{X}_T} = \|u_0\|_{1,B_T} \le C_\gamma \|\lambda\|_{-1/2,\Gamma}$$

it follows that

$$\|L\|_{H^{-1/2}(\Gamma)\to Z} \le C_\gamma \qquad \text{where} \qquad Z \in \{H, V, H^1(B_T), \mathbb{X}_T\}. \tag{7.6}$$

Proposition 7.2.1 (Strong solutions in bounded domain)**.** *Let $\lambda \in \mathcal{C}_0^2([0,\infty); H^{-1/2}(\Gamma))$. Then, there exists (a unique)*

$$u_T : [0,\infty) \to \mathbb{X}_T = H_0^1(B_T) \cap H_\Delta^1(B_T \setminus \Gamma),$$

satisfying

$$\ddot{u}_T(t) = \Delta u_T(t) \qquad \forall t \ge 0, \tag{7.7a}$$

$$[\![\partial_\nu u_T]\!](t) = \lambda(t), \qquad \forall t \ge 0, \tag{7.7b}$$

$$u_T(0) = \dot{u}_T(0) = 0, \tag{7.7c}$$

with regularity

$$u_T \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H_0^1(B_T)) \cap \mathcal{C}([0,\infty); \mathbb{X}_T). \tag{7.8}$$

Moreover, for all $t \ge 0$,

$$\|\nabla u_T(t)\|_{B_T} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)\big), \tag{7.9a}$$

$$\|\Delta u_T(t)\|_{B_T\setminus\Gamma} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)\big), \tag{7.9b}$$

$$\|u_T(t)\|_{1,B_T} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + c(T)\, B(\lambda, t)\big), \tag{7.9c}$$

$$\|\dot{u}_T(t)\|_{1,B_T} \le C_\gamma \big(\|\dot{\lambda}(t)\|_{-1/2,\Gamma} + \sqrt{2}\, B(\lambda, t)\big), \tag{7.9d}$$

$$\|\{\!\{\partial_\nu u_T\}\!\}(t)\|_{-1/2,\Gamma} \le \sqrt{2} C_\gamma C_\nu \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)\big), \tag{7.9e}$$

where

$$c(T) := \sqrt{1 + C_T^2} \quad \textit{and} \quad B(\lambda, t) := \int_0^t \|\lambda(\tau) - \ddot{\lambda}(\tau)\|_{-1/2,\Gamma} \mathrm{d}\tau.$$

Proof. Lifting and decomposition. Consider the functions

$$u_0(t) := L\lambda(t) \qquad f(t) := L(\lambda(t) - \ddot{\lambda}(t)) = u_0(t) - \ddot{u}_0(t)$$

and note that $f : [0,\infty) \to V$ is continuous and that $u_0 \in \mathcal{C}^2_0([0,\infty); \mathbb{X}_T)$ by (7.6). Also by (7.6), it follows that

$$\int_0^t \|f(\tau)\|_{B_T} \mathrm{d}\tau \le C_\gamma B(\lambda, t) \qquad \text{and} \qquad \int_0^t \|\nabla f(\tau)\|_{B_T} \mathrm{d}\tau \le C_\gamma B(\lambda, t). \tag{7.10}$$

The key now is to work with the equation satisfied by $v_0 = u_T - u_0$. Thus, we look for $v_0 : [0,\infty) \to D(A)$ such that

$$\ddot{v}_0(t) = Av_0(t) + f(t) \quad \forall t \ge 0, \qquad v_0(0) = \dot{v}_0(0) = 0. \tag{7.11}$$

By Proposition 6.4.2, equation (7.11) has a solution

$$v_0 \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H^1_0(B_T)) \cap \mathcal{C}([0,\infty); H^1_\Delta(B_T)). \tag{7.12}$$

Moreover, by Proposition 6.4.3 and (7.10), we can bound

$$C_T^{-1}\|v_0(t)\|_{B_T} \le \|\nabla v_0(t)\|_{B_T} \le C_\gamma B(\lambda, t), \tag{7.13a}$$
$$\|\nabla \dot{v}_0(t)\|_{B_T} \le C_\gamma B(\lambda, t), \tag{7.13b}$$
$$\|\dot{v}_0(t)\|_{B_T} \le C_\gamma B(\lambda, t), \tag{7.13c}$$
$$\|\Delta v_0(t)\|_{B_T} \le C_\gamma B(\lambda, t). \tag{7.13d}$$

Recomposition. Consider now the function $u_T(t) := u_0(t)+v_0(t)$. Note that u_T takes values in $\mathbb{X}_T$ for all t and that u_T satisfies (7.8) by (7.12) and the fact that $u_0 \in \mathcal{C}^2([0,\infty); \mathbb{X}_T)$. Also, notice that $u_0(t) = \Delta u_0(t)$ for all t (see the definition of the lifting) and therefore

$$\begin{aligned}
\ddot{u}_T(t) &= \ddot{u}_0(t) + \ddot{v}_0(t)\\
&= \ddot{u}_0(t) - u_0(t) + \Delta u_0(t) + \Delta v_0(t) + f(t)\\
&= \Delta(u_0(t) + v_0(t)) + \ddot{u}_0(t) - u_0(t) + u_0(t) - \ddot{u}_0(t) = \Delta u_T(t), \qquad \forall t \ge 0.
\end{aligned}$$

Finally,

$$[\![\partial_\nu u_T]\!](t) = [\![\partial_\nu u_0]\!](t) = \lambda(t) \qquad \forall t \ge 0$$

and

$$u_T(0) = u_0(0) + v_0(0) = L\lambda(0) = 0 \qquad \dot{u}_T(0) = \dot{u}_0(0) + \dot{v}_0(0) = L\dot{\lambda}(0) = 0.$$

This proves that $u_T : [0,\infty) \to \mathbb{X}_T$ satisfies the problem (7.7). (Uniqueness of solution of this problem follows from uniqueness for the associated initial value problem, which was proved in Proposition 6.3.2 in the abstract frame of Chapter 6.)

Bounds. The bounds for u_T follows from those for $v_0(t)$, i.e. (7.13), from the boundedness properties of the lifting L (7.6) and from the fact that

$$\|\{\!\{\partial_\nu u_T\}\!\}(t)\|_{1/2,\Gamma} \le C_\nu \|u_T(t)\|_{\mathbb{X}_T}$$

(see (7.3)). $\square$

Proposition 7.2.2 (Waiting Time on ∂B_T). *Let $u_T : [0,\infty) \to \mathbb{X}_T$ be the solution of (7.7) for $\lambda \in \mathcal{C}_0^2([0,\infty); H^{-1/2}(\Gamma))$. Then*

$$\partial_T^\nu u_T(t) = 0 \qquad 0 \le t \le T + \delta.$$

Proof. A simple uniqueness argument shows that the value of $\lambda(t)$ for $t > T+\delta$ is not relevant. We thus assume that λ is polynomially bounded at infinity, so that $E\lambda \in \mathrm{TD}(H^{-1/2}(\Gamma))$ by Lemma 7.1.2.

By Lemma 7.1.2, Eu_T is an $\mathbb{X}_T$-valued causal distribution satisfying

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Eu_T) = E\ddot{u}_T = E\Delta u_T = \Delta E u_T$$

and

$$[\![\partial_\nu E u_T]\!] = E[\![\partial_\nu u_T]\!] = E\lambda.$$

These equations hold in the sense of $L^2(B_T \setminus \Gamma)$ and $H^{-1/2}(\Gamma)$-valued distributions respectively. Moreover, the bounds of Proposition 7.2.1 show that u_T grows polynomially in the $\mathbb{X}_T$ norm and therefore $Eu_T \in \mathrm{TD}(\mathbb{X}_T)$.

Consider now the distributions

$$w := Eu_T - (\mathcal{S} * E\lambda)|_{B_T} \qquad \text{and} \qquad \xi := \gamma_T w = -\gamma_T(\mathcal{S} * E\lambda).$$

Note that w is a causal solution of the problem

$$w \in \mathrm{TD}(H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)), \qquad \ddot{w} = \Delta w, \qquad [\![\partial_\nu w]\!] = 0, \qquad \gamma_T w = \xi.$$

(The condition $[\![\gamma w]\!] = 0$ is implicit in the space $H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)$.) We can now use a Laplace transform argument (see exercises) to show that

$$w = \mathcal{M} * \xi \qquad \text{where} \qquad \mathcal{M} \in \mathrm{TD}(\mathcal{B}(H^{1/2}(\partial B_T), H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)).$$

By finite speed of propagation of potentials (Proposition 7.1.1), it follows that $\operatorname{supp} \xi \subset [T+\delta, \infty)$, but then, preservation of causality of causal convolution operators (Proposition 3.2.1) proves that $\operatorname{supp} w \subset [T+\delta,\infty)$. This shows that

$$\operatorname{supp} \partial_T^\nu w \subset [T+\delta, \infty),$$

which combined with the fact that $\operatorname{supp} \partial_T^\nu(\mathcal{S} * E\lambda) \subset [T+\delta,\infty)$ (again by Proposition 7.1.1), implies that

$$\operatorname{supp} \partial_T^\nu E u_T \subset [T+\delta, \infty). \tag{7.14}$$

However, $\partial_T^\nu u_T : [0,\infty) \to H^{-1/2}(\partial B_T)$ is a continuous function (u_T is a continuous $\mathbb{X}_T$-valued function and $\partial_T^\nu : \mathbb{X}_T \to H^{-1/2}(\partial B_T)$ is bounded), and thus (7.14) implies the result. □

Proposition 7.2.3 (Extension to free space). *Let $\lambda \in \mathcal{C}_0^2([0,\infty); H^{-1/2}(\Gamma))$ and u_T be the solution of (7.7). Consider the extension*

$$\underline{u}_T : [0,\infty) \to H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))$$

given by

$$\underline{u}_T(t)|_{B_T} = u_T(t) \qquad \text{and} \qquad \underline{u}_T(t)|_{\mathbb{R}^d\setminus B_T} \equiv 0.$$

Then $\underline{u}_T : [0, T+\delta] \to H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ *is continuous and*

$$\underline{u}_T(t) = (\mathcal{S} * E\lambda)(t) \qquad 0 \le t < T+\delta. \tag{7.15}$$

Proof. It is clear that $\underline{u}_T : [0,\infty) \to H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))$ is continuous and that $\underline{u}_T$ satisfies the equations:

$$\ddot{\underline{u}}_T(t) = \Delta \underline{u}_T(t) \qquad \forall t \ge 0, \tag{7.16a}$$

$$[\![\partial_\nu \underline{u}_T]\!](t) = \lambda(t) \qquad \forall t \ge 0, \tag{7.16b}$$

$$[\![\partial_T^\nu \underline{u}_T]\!](t) = \partial_T^\nu u_T(t) \qquad \forall t \ge 0, \tag{7.16c}$$

$$\underline{u}_T(0) = \dot{\underline{u}}_T(0) = 0. \tag{7.16d}$$

The Laplace operator in (7.16a) is the distributional Laplacian in $\mathbb{R}^d \setminus (\Gamma \cup \partial B_T)$, and the jump operator in (7.16c) corresponds to the jump of normal derivative across ∂B_T. By Lemma 7.1.2 (see the argument in the proof of Proposition 7.2.2), the extension $E\underline{u}_T$ is a causal solution of the wave equation

$$E\underline{u}_T \in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T)) \qquad \tfrac{\mathrm{d}^2}{\mathrm{d}t^2}(E\underline{u}_T) = \Delta E\underline{u}_T$$

and therefore, by Kirchhoff's formula (Proposition 3.5.1), we can represent

$$E\underline{u}_T = \mathcal{S} * [\![\partial_\nu E\underline{u}_T]\!] + \mathcal{S}_{\partial B_T} * [\![\partial_T^\nu E\underline{u}_T]\!] = \mathcal{S} * E\lambda + \mathcal{S}_{\partial B_T} * E\partial_T^\nu u_T,$$

where $\mathcal{S}_{\partial B_T}$ is the distribution that defines the single layer operator stemming from the boundary ∂B_T. By Proposition 7.2.2 and causality of (causal) convolution operators, it follows that

$$\mathrm{supp}\,(E\underline{u}_T - \mathcal{S} * E\lambda) = \mathrm{supp}\,(\mathcal{S}_{\partial B_T} * \partial_T^\nu u_T) \subset [T+\delta, \infty).$$

This proves (7.15). Finally,

$$H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus \Gamma) = \{u \in H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T)) \,:\, [\![\partial_T^\nu u]\!] = 0\}$$

and the different Laplace operators in these two sets (Laplacian in $\mathbb{R}^d \setminus \Gamma$ and in $\mathbb{R}^d \setminus (\Gamma \cup \partial B_T)$ respectively) coincide. The fact that $\underline{u}_T$ satisfies the equations (7.16) and Proposition 7.2.2 prove then that $\underline{u}_T(t) \in H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ for all $t \le T+\delta$ and that $\Delta \underline{u}_T(t) = \underline{\Delta u_T}(t)$ for $t \le T+\delta$. □

The results so far. Let us now start with $\lambda \in \mathrm{TD}(H^{-1/2}(\Gamma))$ such that the restriction $\lambda|_{(0,\infty)}$ is in $\mathcal{C}^2_0([0,\infty); H^{-1/2}(\Gamma))$. In order to observe all potentials and operators at time $t = T$ we use Proposition 7.2.3 with the cut-off at T, and evaluate all functions at $t = T$. We then use the bounds of Proposition 7.2.1 to obtain:

$$\begin{aligned} \|\nabla(\mathcal{S} * \lambda)(T)\|_{\mathbb{R}^d} &= \|\nabla \underline{u}_T(T)\|_{\mathbb{R}^d} && \text{(Proposition 7.2.3)} \\ &= \|\nabla u_T(T)\|_{B_T} && \text{(definition of } \underline{u}_T\text{)} \\ &\le C_\gamma(\|\lambda(T)\|_{-1/2,\Gamma} + B(\lambda, T)\Big). && \text{(Proposition 7.2.1)} \end{aligned}$$

Similarly

$$\begin{aligned}
\|(\mathcal{V} * \lambda)(T)\|_{1/2,\Gamma} &\le C_\gamma \|(\mathcal{S} * \lambda)(T)\|_{1,\mathbb{R}^d} && \text{(trace operator)}\\
&= C_\gamma \|u_T(T)\|_{1,B_T} && \text{(Proposition 7.2.3)}\\
&\le C_\gamma^2 \Big(\|\lambda(T)\|_{-1/2,\Gamma} + c(T)\, B(\lambda, T)\Big), && \text{(Proposition 7.2.1)}
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathcal{K}^t * \lambda)(T)\|_{-1/2,\Gamma} &= \|[\![\partial_\nu u_T]\!](T)\|_{-1/2,\Gamma} && \text{(Proposition 7.2.3)}\\
&\le \sqrt{2}\, C_\nu C_\gamma \Big(\|\lambda(T)\|_{-1/2,\Gamma} + B(\lambda, T)\Big). && \text{(Proposition 7.2.1)}
\end{aligned}$$

Using the regularity part of Proposition 7.2.1 and the identification of the potential $\mathcal{S} * \lambda$ with the extensions $\underline{u}_T$, it is also clear that we have proved that

$$\mathcal{S} * \lambda \in \mathcal{C}^2(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \tag{7.17}$$

and therefore

$$\mathcal{V} * \lambda \in \mathcal{C}^1(\mathbb{R}; H^{1/2}(\Gamma))$$

and

$$\mathcal{K}^t * \lambda \in \mathcal{C}(\mathbb{R}; H^{-1/2}(\Gamma)).$$

Note that (7.17) includes the information about the fact that $\ddot{\underline{u}}_T(0) = \Delta \underline{u}_T(0) = 0$, which allows us to show that the potential is actually in the spaces that appear in (7.17).

Concluding remark. We will come back to these conclusions in Section 7.5, where we will use some density and shift arguments to reduce the continuity requirements on the side of the density λ. We will also get rid of the quantity $c(T)$ by changing the way we accumulate information from the data function λ.

7.3 The Galerkin error operator

Recall that the Galerkin (semidiscrete) equations, seen as an approximation process, look for a causal X_h-valued density λ^h such that

$$\langle \mu^h, \mathcal{V} * (\lambda^h - \lambda) \rangle_\Gamma = 0 \qquad \forall \mu^h \in X_h \tag{7.18}$$

and then build $u^h = \mathcal{S} * \lambda^h$. The error operators map λ to

$$\mathcal{E}^h_\lambda * \lambda = \lambda^h - \lambda \qquad \text{and} \qquad \mathcal{E}^h_u * \lambda = u^h - u = \mathcal{S} * \mathcal{E}^h_\lambda * \lambda.$$

These operators have been studied in the Laplace domain in Section 5.4. As in that section, the analysis is carried out through the potential component ($\lambda \mapsto \mathcal{E}^h_u * \lambda$) and then conclusions are drawn using the fact that $\mathcal{E}^h_\lambda * \lambda = [\![\partial_\nu (\mathcal{E}^h_u * \lambda)]\!]$. Note that $\varepsilon_u := \mathcal{E}^h_u * \lambda$ can be characterized by the uniquely solvable problem

$$\varepsilon_u \in \mathrm{TD}(H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus \Gamma)), \qquad \ddot{\varepsilon}_u = \Delta \varepsilon_u, \qquad \gamma \varepsilon_u \in X_h^\circ, \qquad [\![\partial_\nu \varepsilon_u]\!] + \lambda \in X_h.$$

We cut off with a boundary ∂B_T and look for $\varepsilon_T : [0,\infty) \to H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)$ satisfying

$$\ddot{\varepsilon}_T(t) = \Delta\varepsilon_T(t) \qquad \forall t \geq 0, \tag{7.19a}$$
$$\gamma_T\varepsilon_T(t) = 0 \qquad \forall t \geq 0, \tag{7.19b}$$
$$\gamma\varepsilon_T(t) \in X_h^\circ \qquad \forall t \geq 0, \tag{7.19c}$$
$$[\![\partial_\nu\varepsilon_T]\!](t) + \lambda(t) \in X_h \qquad \forall t \geq 0, \tag{7.19d}$$
$$\varepsilon_T(0) = \dot{\varepsilon}_T(0) = 0. \tag{7.19e}$$

The first condition on Γ, (7.19c), encodes the discrete Galerkin equation (7.18) from the point of view of $u^h - u$. At the same time, (7.19d) is used to impose that the semidiscrete unknown in (7.18) takes values in X_h. As we did in (7.5), the boundary condition on ∂B_T will be directly imposed in the space, by considering

$$\varepsilon_T : [0,\infty) \to \mathbb{X}_T = H^1_0(B_T) \cap H^1_\Delta(B_T \setminus \Gamma).$$

The associated dynamical system. The spaces for the second order equations are suggested by the transmission and boundary conditionsin (7.19). We thus define

$$\begin{aligned} H &:= L^2(B_T), \\ V &:= \{u \in H^1_0(B_T) \,:\, \gamma u \in X_h^\circ\}, \\ D(A) &:= \{u \in V \,:\, \Delta u \in L^2(B_T \setminus \Gamma),\ [\![\partial_\nu u]\!] \in X_h\}, \end{aligned}$$

with

$$\|u\|_H := \|u\|_{B_T}, \qquad [u,v] := (\nabla u, \nabla v)_{B_T}.$$

We take A to be the distributional Laplacian in $B_T \setminus \Gamma$. Note that if $u \in H^1_\Delta(B_T \setminus \Gamma)$, then

$$[\![\partial_\nu u]\!] \in X_h \qquad \Longleftrightarrow \qquad (\nabla u, \nabla v)_{B_T\setminus\Gamma} + (\Delta u, v)_{B_T\setminus\Gamma} = 0 \quad \forall v \in V. \tag{7.20}$$

We now verify all the hypotheses in our checklist:

- The fact that $\mathcal{D}(B_T \setminus \Gamma) \subset V \subset H^1(B_T) \subset H$ implies the density and the compactness of the embedding of V in H.

- The constant to relate the norms of H and V is the one of the Poincaré inequality (7.1) and thus $C_\circ = C_T$, with C_T given by (7.1).

- The Green Idendity is just another way of phrasing (7.20).

- Finally, if $f \in H = L^2(B_T)$, we can solve the coercive problem

  $$\left[\begin{array}{l} u \in V, \\ (\nabla u, \nabla v)_{B_T} + (u,v)_{B_T} = (f,v)_{B_T} \qquad \forall v \in V, \end{array}\right.$$

  and then verify that $-\Delta u + u = f$ in $B_T \setminus \Gamma$ (test with a general element of $\mathcal{D}(B_T \setminus \Gamma)$) and then

  $$(\nabla u, \nabla v)_{B_T} + (\Delta u, v)_{B_T\setminus\Gamma} = (\nabla u, \nabla v)_{B_T} + (u,v)_{B_T} - (f,v)_{B_T} = 0 \qquad \forall v \in V,$$

  which proves that $u \in D(A)$.

The lifting. We now define $L : H^{-1/2}(\Gamma) \to V$ through the solution of the problem

$$\left[\begin{array}{l} u_0 \in V, \\ (\nabla u_0, \nabla v)_{B_T} + (u_0, v)_{B_T} = \langle \lambda, \gamma v \rangle_\Gamma \qquad \forall v \in V, \end{array}\right. \tag{7.21}$$

which is the variational formulation of

$$u_0 \in H_0^1(B_T), \tag{7.22a}$$
$$-\Delta u_0 + u_0 = 0 \qquad \text{in } B_T \setminus \Gamma, \tag{7.22b}$$
$$\gamma u_0 \in X_h^\circ, \tag{7.22c}$$
$$[\![\partial_\nu u_0]\!] - \lambda \in X_h. \tag{7.22d}$$

Testing equations (7.21) with $v = u_0$ we can bound

$$\|u_0\|_{\mathbb{X}_T} = \|u_0\|_{1,B_T} \le C_\gamma \|\lambda\|_{-1/2,\Gamma}$$

which yields the estimates

$$\|L\|_{H^{-1/2}(\Gamma)\to Z} \le C_\gamma \qquad \text{where} \qquad Z \in \{H, V, H^1(B_T), \mathbb{X}_T\}. \tag{7.23}$$

Proposition 7.3.1 (Strong solutions in bounded domain)**.** *Let* $\lambda \in \mathcal{C}_0^2([0,\infty); H^{-1/2}(\Gamma))$. *Then, there exists (a unique)*

$$\varepsilon_T : [0,\infty) \to \mathbb{X}_T = H_0^1(B_T) \cap H_\Delta^1(B_T \setminus \Gamma),$$

satisfying

$$\ddot{\varepsilon}_T(t) = \Delta \varepsilon_T(t) \qquad \forall t \ge 0, \tag{7.24a}$$
$$\gamma \varepsilon_T(t) \in X_h^\circ \qquad \forall t \ge 0, \tag{7.24b}$$
$$[\![\partial_\nu \varepsilon_T]\!](t) + \lambda(t) \in X_h \qquad \forall t \ge 0, \tag{7.24c}$$
$$\varepsilon_T(0) = \dot{\varepsilon}_T(0) = 0, \tag{7.24d}$$

with regularity

$$\varepsilon_T \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H_0^1(B_T)) \cap \mathcal{C}([0,\infty); \mathbb{X}_T). \tag{7.25}$$

Moreover, for all $t \ge 0$,

$$\|\nabla \varepsilon_T(t)\|_{B_T} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda,t)\big), \tag{7.26a}$$
$$\|\Delta \varepsilon_T(t)\|_{B_T\setminus\Gamma} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda,t)\big), \tag{7.26b}$$
$$\|\varepsilon_T(t)\|_{1,B_T} \le C_\gamma \big(\|\lambda(t)\|_{-1/2,\Gamma} + c(T)\, B(\lambda,t)\big), \tag{7.26c}$$
$$\|\dot{\varepsilon}_T(t)\|_{1,B_T} \le C_\gamma \big(\|\dot{\lambda}(t)\|_{-1/2,\Gamma} + \sqrt{2}\, B(\lambda,t)\big), \tag{7.26d}$$
$$\|[\![\partial_\nu \varepsilon_T]\!](t)\|_{-1/2,\Gamma} \le \sqrt{2} C_\gamma C_\nu \big(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda,t)\big), \tag{7.26e}$$

where

$$c(T) := \sqrt{1 + C_T^2} \quad \textit{and} \quad B(\lambda,t) := \int_0^t \|\lambda(\tau) - \ddot{\lambda}(\tau)\|_{-1/2,\Gamma} \mathrm{d}\tau.$$

Proof. The proof is very similar to that of Proposition 7.2.1. We start by defining

$$u_0(t) := -L\lambda(t), \qquad f(t) := u_0(t) - \ddot{u}_0(t) = L(\ddot{\lambda}(t) - \lambda(t)),$$

consider then the solution $v_0 : [0, \infty) \to D(A)$ of the problem

$$\ddot{v}_0(t) = Av_0(t) + f(t) \quad \forall t \geq 0, \qquad v_0(0) = \dot{v}_0(0) = 0, \tag{7.27}$$

and finally propose $\varepsilon_T := u_0 + v_0$. The key facts are the continuity of $f : [0, \infty) \to V$ (which means that (7.27) has strong solutions in the sense of Section 6.4) and the equality $\Delta u_0(t) = u_0(t)$ for all t.

Since $L\lambda(t) \in V$ and $v_0(t) \in D(A) \subset V$ for all $t \geq 0$, it is clear that $\varepsilon_T(t) \in V$ for all t. To verify the other transmission condition note that

$$[\![\partial_\nu \varepsilon_T]\!](t) + \lambda(t) = \underbrace{-[\![\partial_\nu L\lambda(t)]\!] + \lambda(t)}_{\text{(see (7.22))}} + \underbrace{[\![\partial_\nu v_0(t)]\!]}_{(v_0(t) \in D(A))} \in X_h.$$

With this we show that ε_T satisfies equations (7.24). Using the fact that $u_0 \in \mathcal{C}^2([0, \infty); \mathbb{X}_T)$ and Proposition 6.4.2 we can prove the desired regularity (7.25).

We can finally use Proposition 6.4.3 to bound v_0 in terms of f, as well as (7.23) to bound f in terms of λ. This leads to the estimates

$$\begin{aligned} C_T^{-1} \|v_0(t)\|_{B_T} \leq \|\nabla v_0(t)\|_{B_T} &\leq C_\gamma B(\lambda, t), \\ \|\nabla \dot{v}_0(t)\|_{B_T} &\leq C_\gamma B(\lambda, t), \\ \|\dot{v}_0(t)\|_{B_T} &\leq C_\gamma B(\lambda, t), \\ \|\Delta v_0(t)\|_{B_T \setminus \Gamma} &\leq C_\gamma B(\lambda, t). \end{aligned}$$

Finally, (7.23) is used again to bound u_0 and this finishes the proof. □

Proposition 7.3.2 (Waiting Time on ∂B_T)**.** *Let $\varepsilon_T : [0, \infty) \to \mathbb{X}_T$ be the solution of (7.24) for $\lambda \in \mathcal{C}_0^2([0, \infty); H^{-1/2}(\Gamma))$. Then*

$$\partial_T^\nu \varepsilon_T(t) = 0 \qquad \forall t \leq T + \delta.$$

Proof. We can restrict our attention to densities λ with polynomial growth in the time variable, which imples that ε_T has also polynomial growth (see the bounds in Proposition 7.3.1). We then define the continuous function $[\![\partial_\nu \varepsilon_T]\!] : [0, \infty) \to H^{-1/2}(\Gamma)$. What is left of the proof consists of a straightforward adaption of the proof of Proposition 7.2.2 in order to first show that

$$\operatorname{supp} \big(E\varepsilon_T - (\mathcal{S} * E[\![\partial_\nu \varepsilon_T]\!])|_{B_T} \big) \subset [T + \delta, \infty)$$

via a causality argument and then by finite speed of propagation of potentials (Proposition 7.1.1), and to prove then that

$$\operatorname{supp} \partial_T^\nu E\varepsilon_T \subset [T + \delta, \infty).$$

Details are left to the reader. □

Proposition 7.3.3 (Recovery of the density)**.** *Let $\varepsilon_T : [0,\infty) \to \mathbb{X}_T$ be the solution of (7.24) for $\lambda \in \mathcal{C}^2_0([0,\infty); H^{-1/2}(\Gamma))$. Consider the function*

$$\varepsilon_\lambda : [0,\infty) \to H^{-1/2}(\Gamma), \qquad \varepsilon_\lambda(T) := [\![\partial_\nu \varepsilon_T]\!](T) \qquad \forall T \geq 0.$$

Then $\varepsilon_\lambda \in \mathcal{C}([0,\infty); H^{-1/2}(\Gamma))$ and

$$\varepsilon_\lambda(t) = [\![\partial_\nu \varepsilon_T]\!](t) \qquad t \leq T + \delta.$$

Proof. A new causality argument can be invoked (see exercises) to show that

$$\varepsilon_T(t) = \varepsilon_{T+M}(t)|_{B_T} \qquad \forall t \leq T + \delta \qquad M \geq 0.$$

Therefore

$$[\![\partial_\nu \varepsilon_T]\!](T) = [\![\partial_\nu \varepsilon_{T+M}]\!](T) \qquad \forall M \geq 0.$$

Finally, this implies that

$$\varepsilon_\lambda(t) = [\![\partial_\nu \varepsilon_t]\!](t) = [\![\partial_\nu \varepsilon_T]\!](t) \qquad \forall t \in [0,T].$$

Since $\varepsilon_T : [0,\infty) \to \mathbb{X}_T$ is continuous (Proposition 7.3.1) and $[\![\partial_\nu \cdot\,]\!] : \mathbb{X}_T \to H^{-1/2}(\Gamma)$ is bounded, it follows that $\varepsilon_\lambda : [0,T] \to H^{-1/2}(\Gamma)$ is continuous for all T. □

Proposition 7.3.4 (Extension to free space)**.** *Let $\lambda \in \mathcal{C}^2_0([0,\infty); X)$, ε_T be the solution of (7.24) and ε_λ be defined by Proposition 7.3.3. Consider the extension*

$$\underline{\varepsilon}_T : [0,\infty) \to H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T)) \qquad \underline{\varepsilon}_T(t)|_{B_T} = \varepsilon_T(t), \quad \underline{\varepsilon}_T(t)|_{\mathbb{R}^d \setminus B_T} \equiv 0$$

Then $\underline{\varepsilon}_T : [0, T+\delta] \to H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ is continuous and

$$\underline{\varepsilon}_T(t) = (\mathcal{S} * E\varepsilon_\lambda)(t) \qquad 0 \leq t < T + \delta. \tag{7.28}$$

Proof. It is clear that $\underline{\varepsilon}_T : [0,\infty) \to H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))$ is continuous. Let then

$$\mu := [\![\partial_\nu \varepsilon_T]\!] : [0,\infty) \to H^{-1/2}(\Gamma), \qquad \mu_T := \partial^\nu_T \varepsilon_T : [0,\infty) \to H^{-1/2}(\partial B_T),$$

and note that by Proposition 7.3.3, $\mu(t) = \varepsilon_\lambda(t)$ for $t \leq T + \delta$, and that $\underline{u}_T$ satisfies the equations:

$$\begin{aligned} \ddot{\underline{\varepsilon}}_T(t) &= \Delta \underline{\varepsilon}_T(t) && \forall t \geq 0, && (7.29a)\\ [\![\partial_\nu \underline{\varepsilon}_T]\!](t) &= \mu(t) && \forall t \geq 0, && (7.29b)\\ [\![\partial^\nu_T \underline{\varepsilon}_T]\!](t) &= \mu_T(t) && \forall t \geq 0, && (7.29c)\\ \underline{\varepsilon}_T(0) &= \dot{\underline{\varepsilon}}_T(0) = 0. && && (7.29d) \end{aligned}$$

Proceeding as in the proof of Proposition 7.2.3, we can write

$$E\underline{\varepsilon}_T = \mathcal{S} * E\mu + \mathcal{S}_{\partial B_T} * E\mu_T = \mathcal{S} * E\varepsilon_\lambda + \mathcal{S} * E(\mu - \varepsilon_\lambda) + \mathcal{S}_{\partial B_T} * E\mu_T.$$

Since $\operatorname{supp} E\mu_T \subset [T+\delta,\infty)$ (by the Waiting Time Property) and $\operatorname{supp} E(\mu - \varepsilon_\lambda) \subset [T+\delta,\infty)$, the result follows readily. □

Corollary 7.3.5 (Identification of the Galerkin error operators)**.** *In the conditions and notation of Proposition 7.3.4 it follows that*

$$E\varepsilon_\lambda = \mathcal{E}^h_\lambda * E\lambda,$$

and therefore

$$\varepsilon_\lambda(t) = (\mathcal{E}^h_\lambda * E\lambda)(t) \qquad \forall t \geq 0,$$

and

$$\underline{\varepsilon}_T(t) = (\mathcal{E}^h_u * E\lambda)(t) \qquad 0 \leq t < T + \delta.$$

Proof. By Proposition 7.3.3,

$$\varepsilon_\lambda(t) + \lambda(t) = [\![\partial_\nu \varepsilon_t]\!](t) + \lambda(t) \in X_h \qquad \forall t \geq 0.$$

On the other hand, since $\gamma\varepsilon_T(t) \in X_h^\circ$ for all t, by Proposition 7.3.4,

$$(\mathcal{V} * E\varepsilon_\lambda)(t) = \gamma(\mathcal{S} * E\varepsilon_\lambda)(t) = \gamma\underline{\varepsilon}_t(t) \in X_h^\circ.$$

This shows that $E\varepsilon_\lambda = \mathcal{E}^h_\lambda * E\lambda$. □

Conclusions. Like we did in Section 7.2, this collection of results can be gathered in some regularity statements plus some estimates. For a causal distribution λ such that $\lambda|_{(0,\infty)} \in \mathcal{C}^2_0(\mathbb{R}; H^{-1/2}(\Gamma))$, we thus prove that

$$\mathcal{E}^h_u * \lambda \in \mathcal{C}^2(\mathbb{R}; L^2(B_T)) \cap \mathcal{C}^1(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \tag{7.30}$$

and (therefore)

$$\mathcal{E}^h_\lambda * \lambda \in \mathcal{C}(\mathbb{R}; H^{-1/2}(\Gamma)).$$

In addition to this, we get the estimates:

$$\|\nabla(\mathcal{E}^h_u * \lambda)(t)\|_{\mathbb{R}^d} \leq C_\gamma(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)), \tag{7.31a}$$

$$\|(\mathcal{E}^h_u * \lambda)(t)\|_{1,\mathbb{R}^d} \leq C_\gamma(\|\lambda(t)\|_{-1/2,\Gamma} + c(t)\, B(\lambda, t)), \tag{7.31b}$$

$$\|(\mathcal{E}^h_\lambda * \lambda)(t)\|_{-1/2,\Gamma} \leq \sqrt{2} C_\nu C_\gamma(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)). \tag{7.31c}$$

We will come back to these results in Section 7.5.

7.4 The Galerkin solver

The Galerkin equations, seen from the point of view of data, look for a causal X_h-valued density λ^h such that

$$\langle \mu^h, \mathcal{V} * \lambda^h \rangle_\Gamma = \langle \mu^h, \beta \rangle_\Gamma \qquad \forall \mu^h \in X_h,$$

and then build $u^h = \mathcal{S} * \lambda^h$. The corresponding operators are

$$\mathcal{G}^h_\lambda * \beta = \lambda^h \qquad \text{and} \qquad \mathcal{G}^h_u * \beta = u^h = \mathcal{S} * \mathcal{G}^h_\lambda \beta.$$

These operators were studied in the Laplace domain in Section 5.3. Once again, we will study the operator $\beta \mapsto \mathcal{G}_u^h * \beta$ and then use that $\mathcal{G}_\lambda^h * \beta = [\![\partial_\nu(\mathcal{G}_u^h * \beta)]\!]$. The distribution $u^h = \mathcal{G}_u^h * \beta$ is characterized as the solution of the evolution problem

$$u^h \in \mathrm{TD}(H^1(\mathbb{R}^d) \cap H^1_\Delta(\mathbb{R}^d \setminus \Gamma)), \qquad \ddot{u}^h = \Delta u^h, \qquad \gamma u^h - \beta \in X_h^\circ, \qquad [\![\partial_\nu u^h]\!] \in X_h.$$

The process is triggered again with a cut-off on ∂B_T, so that we look for $u_T^h : [0, \infty) \to H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma)$

$$\begin{aligned} \ddot{u}_T^h(t) &= \Delta u_T^h(t) && \forall t \geq 0, && (7.32a)\\ \gamma_T u_T^h(t) &= 0 && \forall t \geq 0, && (7.32b)\\ \gamma u_T^h(t) - \beta(t) &\in X_h^\circ && \forall t \geq 0, && (7.32c)\\ [\![\partial_\nu u_T^h]\!](t) &\in X_h && \forall t \geq 0, && (7.32d)\\ u_T^h(0) = \dot{u}_T^h(0) &= 0. && && (7.32e) \end{aligned}$$

This problem uses the same set of transmission conditions as the one satisfied by the Galerkin error operator (7.19), with the main difference that the essential transmission condition is non-homogeneous while the natural one is homogeneous. As usual, the homogeneous boundary condition will be implicitly given by looking for $u_T^h : [0, \infty) \to \mathbb{X}_T$.

The associated dynamical system. It will be no surprise to the reader that, given the coincidences between (7.19) and (7.20), the functional setting for this section is exactly the same as that of Section 7.3.

The lifting. Let $L : H^{1/2} \to H_0^1(B_T) \neq V$ be defined with the variational equations

$$\left[\begin{array}{l} u_0 \in H_0^1(B_T), \quad \gamma u - \beta \in X_h^\circ, \\ (\nabla u_0, \nabla v)_{B_T} + (u_0, v)_{B_T} = 0 \quad \forall v \in V, \end{array}\right. \tag{7.33}$$

or equivalently, with the transmission problem

$$\begin{aligned} u_0 &\in H_0^1(B_T), && && (7.34a)\\ -\Delta u_0 + u_0 &= 0 && \text{in } B_T \setminus \Gamma, && (7.34b)\\ \gamma u_0 - \beta &\in X_h^\circ, && && (7.34c)\\ [\![\partial_\nu u_0]\!] &\in X_h. && && (7.34d) \end{aligned}$$

Testing equations (7.33) with $v = u_0 - \gamma^\dagger \beta$ (recall the lifting of the trace $\gamma^\dagger$ in (7.4)), we can bound

$$\|u_0\|_{\mathbb{X}_T} = \|u_0\|_{1,B_T} \leq \|\gamma^\dagger \beta\|_{1,B_T} \leq C_\gamma^\dagger \|\beta\|_{1/2,\Gamma},$$

and thus

$$\|L\|_{H^{1/2}(\Gamma) \to Z} \leq C_\gamma^\dagger \qquad \text{where} \qquad Z \in \{H, H^1(B_T), \mathbb{X}_T\}. \tag{7.35}$$

Proposition 7.4.1 (Weak solutions in bounded domain)**.** *Let $\beta \in \mathcal{C}_0^2([0,\infty); H^{1/2}(\Gamma))$. Then, there exists (a unique)*

$$u_T^h : [0,\infty) \to H_0^1(B_T),$$

satisfying

$$\begin{aligned}
\langle \ddot{u}_T^h(t), v\rangle_{V'\times V} + (\nabla u_T^h(t), \nabla v)_{B_T} &= 0 \qquad \forall v \in V \qquad \forall t \geq 0, && (7.36a)\\
\gamma u_T^h(t) - \beta(t) &\in X_h^\circ \qquad\qquad\qquad \forall t \geq 0, && (7.36b)\\
u_T^h(0) = \dot{u}_T^h(0) &= 0, && (7.36c)
\end{aligned}$$

with regularity

$$u_T^h \in \mathcal{C}^2([0,\infty); V') \cap \mathcal{C}^1([0,\infty); L^2(B_T)) \cap \mathcal{C}([0,\infty); H_0^1(B_T)). \tag{7.37}$$

Moreover, for all $t \geq 0$,

$$\begin{aligned}
\|\nabla u_T^h(t)\|_{B_T} &\leq C_\gamma^\dagger\big(\|\beta(t)\|_{1/2,\Gamma} + B(\beta,t)\big),\\
\|u_T^h(t)\|_{1,B_T} &\leq C_\gamma^\dagger\big(\|\beta(t)\|_{1/2,\Gamma} + c(T)\, B(\beta,t)\big),\\
\|\dot{u}_T^h(t)\|_{B_T} &\leq C_\gamma^\dagger\big(\|\beta(t)\|_{1/2,\Gamma} + B(\beta,t)\big),
\end{aligned}$$

where

$$c(T) := \sqrt{1 + C_T^2} \quad \text{and} \quad B(\beta,t) := \int_0^t \|\beta(\tau) - \ddot{\beta}(\tau)\|_{1/2,\Gamma} \mathrm{d}\tau.$$

Proof. Let

$$u_0(t) := L\beta(t), \qquad f(t) := u_0(t) - \ddot{u}_0(t) = L(\beta(t) - \ddot{\beta}(t))$$

and note that $u_0 \in \mathcal{C}_0^2([0,\infty)); \mathbb{X}_T)$ and $f \in \mathcal{C}([0,\infty); H)$. However, f does not take values in V. We then consider the solution $v_0 : [0,\infty) \to V$ to the problem

$$\begin{aligned}
&\langle \ddot{v}_0(t), v\rangle_{V'\times V} + (\nabla v_0(t), \nabla v)_{B_T} = (f(t), v)_{B_T} \quad \forall v \in V \qquad \forall t \geq 0, && (7.38a)\\
&v_0(0) = \dot{v}_0(0) = 0. && (7.38b)
\end{aligned}$$

By Proposition 6.5.1,

$$v_0 \in \mathcal{C}^1([0,\infty); L^2(B_T)) \cap \mathcal{C}([0,\infty); H^1(B_T))$$

and

$$C_T^{-1}\|v_0(t)\|_{B_T} \leq \|\nabla v_0(t)\|_{B_T} \leq \int_0^t \|f(\tau)\|_{B_T} \mathrm{d}\tau \leq C_\gamma^\dagger B(\beta,t),$$

where we have used (7.35) in the last inequality. We can thus define $u_T^h := u_0 + v_0$. What is left is the proof that u_T^h satisfies equations (7.36), since everything else follows readily from what has been shown for v_0 and from the properties of the lifting L (7.35). To show that u_T^h satisfies (7.36) we first note that

$$\gamma u_T^h(t) - \beta(t) = \underbrace{\gamma L\beta(t) - \beta(t)}_{\text{(see (7.33))}} + \underbrace{\gamma v_0(t)}_{(v_0(t)\in V)} \in X_h^\circ.$$

Also, using the fact that $\langle\cdot,\cdot\rangle_{V'\times V}$ extends $(\cdot,\cdot)_H$, we prove that

$$\begin{aligned}
\langle \ddot{u}^h_T(t), v\rangle_{V'\times V} &+ (\nabla u^h_T(t), \nabla v)_V \\
&= (\ddot{u}_0(t), v)_{B_T} + (\nabla u_0(t), \nabla v)_{B_T} + (f(t), v)_V && \text{(by (7.38a))}\\
&= (u_0(t), v)_{B_T} + (\nabla u_0(t), \nabla v)_{B_T} && \text{(definition of } f)\\
&= 0 \qquad \forall v \in V. && \text{(by (7.35), } u_0 = L\beta)
\end{aligned}$$

This finishes the proof. $\square$

Proposition 7.4.2 (Strong solutions in bounded domain). *Let $\beta \in \mathcal{C}^3_0([0,\infty); H^{1/2}(\Gamma))$. Then, there exists (a unique)*

$$u^h_T : [0,\infty) \to \mathbb{X}_T = H^1_0(B_T) \cap H^1_\Delta(B_T \setminus \Gamma),$$

satisfying

$$\ddot{u}^h_T(t) = \Delta u^h_T(t) \qquad \forall t \geq 0, \tag{7.39a}$$

$$\gamma u^h_T(t) - \beta(t) \in X^\circ_h \qquad \forall t \geq 0, \tag{7.39b}$$

$$[\![\partial_\nu u^h_T]\!](t) \in X_h \qquad \forall t \geq 0, \tag{7.39c}$$

$$u^h_T(0) = \dot{u}^h_T(0) = 0, \tag{7.39d}$$

with regularity

$$u^h_T \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H^1_0(B_T)) \cap \mathcal{C}([0,\infty); \mathbb{X}_T). \tag{7.40}$$

Moreover, u^h_T is also the solution of (7.36) *and*

$$\begin{aligned}
\|\Delta u^h_T(t)\|_{B_T\setminus\Gamma} &\leq C^\dagger_\gamma(\|\beta(t)\|_{1/2,\Gamma} + 2B(\dot\beta, t)),\\
\|\nabla \dot{u}^h_T(t)\|_{B_T\setminus\Gamma} &\leq C^\dagger_\gamma(\|\dot\beta(t)\|_{1/2,\Gamma} + B(\dot\beta, t)),\\
\|[\![\partial_\nu u^h_T]\!](t)\|_{-1/2,\Gamma} &\leq \sqrt{2}C_\nu C^\dagger_\gamma(2\|\beta(t)\|_{1/2,\Gamma} + B(\beta, t) + 2B(\dot\beta, t)).
\end{aligned}$$

Proof. We define $u_0 \in \mathcal{C}^3_0([0,\infty); \mathbb{X}_T)$ and $f \in \mathcal{C}^1_0([0,\infty); L^2(B_T))$ as in the proof of Proposition 7.4.1:

$$u_0(t) := L\beta(t), \qquad f(t) := u_0(t) - \ddot{u}_0(t).$$

Then we consider the solution $v_0 : [0,\infty) \to D(A)$ of the problem

$$\ddot{v}_0(t) = \Delta v_0(t) + f(t) \qquad t \geq 0, \qquad v_0(0) = \dot{v}_0(0) = 0,$$

and we construct $u^h_T := u_0 + v_0$. It is clear that $u^h_T(0) = \dot{u}^h_T(0) = 0$. Also,

$$\gamma u^h_T(t) - \beta(t) = \underbrace{\gamma L\beta(t) - \beta(t)}_{\text{(see (7.33))}} + \underbrace{\gamma v_0(t)}_{(v_0(t)\in D(A)\subset V)} \in X^\circ_h$$

and

$$[\![\partial_\nu u^h_T]\!](t) = \underbrace{[\![\partial_\nu u_0]\!](t)}_{\text{(see (7.34))}} + \underbrace{[\![\partial_\nu v_0]\!](t)}_{(v_0(t)\in D(A))} \in X_h.$$

The right hand side of the problem satisfied by v_0 is designed so that u_T^h satisfies the differential equation

$$\ddot{u}_T^h(t) = \Delta u_T^h(t) \qquad t \geq 0,$$

and therefore u_T^h is the solution of (7.39). It follows from Proposition 6.5.2 that u_T^h satisfies (7.40). The bound for $\|\Delta u_T^h(t)\|_{B_T\setminus\Gamma}$ follows then from (7.35) and Proposition 6.5.2. This bound and the bound for $\|\nabla u_T^h(t)\|_{B_T}$ given in Proposition 7.4.1 provide then a proof of the bound for $\|[\![\partial_\nu u_T^h]\!](t)\|_{-1/2,\Gamma}$, which finishes the proof. $\square$

The final steps. Identification of the Galerkin solver from extensions of the functions u_T^h follows exactly the same strategy as in Section 7.3, with very much the same proofs. Here is a simple sketch of the process.

- Comparing Eu_T^h with $(\mathcal{S} * E[\![\partial_\nu u_T^h]\!])|_{B_T}$ (see Propositions 7.2.2 and 7.3.2) we prove that
$$\partial_T^\nu u_T^h(t) = 0 \qquad 0 \leq t \leq T + \delta.$$

- Doing as in Proposition 7.3.3 we define
$$\lambda^h : [0, \infty) \to H^{-1/2}(\Gamma), \qquad \lambda^h(T) := [\![\partial_\nu u_T^h]\!](T) \quad T \geq 0 \tag{7.41}$$
and show that $\lambda^h(t) = [\![\partial_\nu u_T^h]\!](t)$ for all $t \in [0, T + \delta]$. This implies that $\lambda^h \in \mathcal{C}([0, \infty); H^{-1/2}(\Gamma))$.

- With the same proof as in Proposition 7.3.4, we next show that
$$\underline{u}_T^h(t) = (\mathcal{S} * E\lambda^h)(t) \qquad 0 \leq t \leq T + \delta,$$
and that $\underline{u}_T^h(t) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ for all $t \leq T + \delta$. (As usual, the underlining makres reference to extension by zero to $\mathbb{R}^d \setminus B_T$.)

All of this leads to the final goal. We defer the collection of bounds for the Galerkin solvers to the next section.

Proposition 7.4.3 (Identification of the Galerkin solver)**.** *Let u_T^h be the functions given by Proposition 7.4.2 and let λ^h be defined by* (7.41)*. Then*

$$E\lambda^h = \mathcal{G}_\lambda^h * E\beta$$

and therefore

$$\lambda^h(t) = (\mathcal{G}_\lambda^h * \beta)(t) \qquad \forall t \geq 0,$$

and

$$\underline{u}_T^h(t) = (\mathcal{G}_u^h * E\beta)(t) \qquad 0 \leq t < T + \delta.$$

7.5 Final version of all the results

In this section, we look carefully to everything we obtained in Sections 7.2, 7.3 and 7.4 and refine the results so that they hold for weaker densities or right hand sides. Some ingredients first:

- We will use spaces of the form

$$\begin{aligned}\mathcal{W}^k_+(\mathbb{R};X) &:= \{\beta\in\mathcal{C}^{k-1}(\mathbb{R};X)\,:\,\operatorname{supp}\beta\subset[0,\infty),\quad \beta^{(k)}\in L^1_{\mathrm{loc}}(\mathbb{R};X)\},\\ \mathcal{C}^k_+(\mathbb{R};X) &:= \{\beta\in\mathcal{C}^{k}(\mathbb{R};X)\,:\,\operatorname{supp}\beta\subset[0,\infty)\}.\end{aligned}$$

- A generic constant $C = C(\Gamma)$ will be used to collect all constants $C_\gamma, C_\gamma^\dagger, C_\nu$, etc, that appeared before. We will also consider a generic function $c(t)\le C_1 + C_2\,t$ to display linear growth of constants at infinity.

- So far, dependence of operators with respect to data has been displayed using the growing norms

$$B(\psi,t) := \int_0^t \|\psi(\tau)-\ddot\psi(\tau)\|\mathrm{d}\tau,$$

  where the norm that we are integrating was not explicited. We will present the results with respect to the growing norms

$$H_k(\psi,t\,|\,X) := \sum_{\ell=0}^k \int_0^t \|\psi^{(\ell)}(\tau)\|_X\,\mathrm{d}\tau.$$

  Note that if $\psi\in\mathcal{W}^1_+(\mathbb{R};X)$, then

$$\|\psi(t)\|_X = \left\|\int_0^t \dot\psi(\tau)\mathrm{d}\tau\right\|_X \le H_1(\psi,t\,|\,X).$$

- Finally we consider the operator

$$\partial^{-1}u(t) := \int_0^t u(\tau)\mathrm{d}\tau,$$

  i.e., $\mathcal{L}\{\partial^{-1}u\} = s^{-1}\mathrm{U}(s)$.

Theorem 7.5.1 (Mapping properties for the single layer potential)**.** *If $\lambda\in\mathcal{W}^1_+(\mathbb{R};H^{-1/2}(\Gamma))$, then*

$$\mathcal{S}*\lambda\in\mathcal{C}^1_+(\mathbb{R};L^2(\mathbb{R}^d))\cap\mathcal{C}_+(\mathbb{R};H^1(\mathbb{R}^d)) \tag{7.42}$$

and therefore

$$\mathcal{V}*\lambda\in\mathcal{C}_+(\mathbb{R};H^{1/2}(\Gamma)).$$

Moreover, for all $t\ge 0$

$$\begin{aligned}\|(\mathcal{S}*\lambda)(t)\|_{1,\mathbb{R}^d} &\le C\,H_2(\partial^{-1}\lambda,t\,|\,H^{-1/2}(\Gamma)), &(7.43a)\\ \|(\mathcal{V}*\lambda)(t)\|_{1/2,\Gamma} &\le C\,H_2(\partial^{-1}\lambda,t\,|\,H^{-1/2}(\Gamma)). &(7.43b)\end{aligned}$$

If additionally $\lambda \in \mathcal{W}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$*, then*

$$\mathcal{K}^t * \lambda \in \mathcal{C}_+(\mathbb{R}; H^{-1/2}(\Gamma)). \tag{7.44}$$

and for all $t \geq 0$

$$\|(\mathcal{K}^t * \lambda)(t)\|_{-1/2,\Gamma} \leq C\, H_2(\lambda, t \,|\, H^{-1/2}(\Gamma)). \tag{7.45}$$

Proof. For causal $\lambda \in \mathcal{C}^2(\mathbb{R}; H^{-1/2}(\Gamma))$, we let $u := \mathcal{S} * \lambda$. From the results of Section 7.2 (see specifically Propositions 7.2.1 and 7.2.3, as well as the conclusions drawn from the latter at the end of the section), it follows that

$$u \in \mathcal{C}^2_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1_+(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$$

and

$$\begin{aligned} \|u(t)\|_{1,\mathbb{R}^d} &\leq C(\|\lambda(t)\|_{-1/2,\Gamma} + c(t)B(\lambda, t)), && (7.46a)\\ \|\dot u(t)\|_{1,\mathbb{R}^d} &\leq C(\|\dot\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)), && (7.46b)\\ \|\ddot u(t)\|_{\mathbb{R}^d\setminus\Gamma} + \|\{\!\{\partial_\nu u\}\!\}(t)\|_{-1/2,\Gamma} &\leq C(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)). && (7.46c) \end{aligned}$$

(Note that the equation $\ddot u(t) = \Delta u(t)$ provides the bound for the second derivative.) We now endow the space

$$X_{\rm aux} := \{\lambda \in \mathcal{C}^2([0,T]; H^{-1/2}(\Gamma)) \,:\, \lambda(0) = \dot\lambda(0) = 0\},$$

with the norm)

$$H_2(\lambda, T \,|\, H^{-1/2}(\Gamma)) = \sum_{\ell=0}^{2} \int_0^T \|\lambda^{(\ell)}(\tau)\|_{-1/2,\Gamma} \mathrm{d}\tau$$

and consider the space

$$\begin{aligned} Y_{\rm aux} \;:=\; &\{u \in \mathcal{C}^2([0,T]; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T]; H^1(\mathbb{R}^d)) \cap \mathcal{C}([0,T]; H^1_\Delta(\mathbb{R}^d \setminus \Gamma))\\ &\quad : u(0) = \dot u(0) = \ddot u(0) = 0\}, \end{aligned}$$

(i.e., elements of $Y_{\rm aux}$ can be extended by zero to $(-\infty, T]$ without losing continuity), endowed with the norm

$$\max_{0\leq t\leq T} \|\ddot u(t)\|_{\mathbb{R}^d} + \max_{0\leq t\leq T} \|\nabla \dot u(t)\|_{\mathbb{R}^d} + \max_{0\leq t\leq T} \|\Delta u(t)\|_{\mathbb{R}^d\setminus\Gamma}.$$

Note that $Y_{\rm aux}$ is a Banach space. Looking at values in the interval $[0,T]$, the bounds (7.44) provide the continuity of the map $X_{\rm aux} \ni \lambda \mapsto u \in Y_{\rm aux}$. We can thus extend this map to the closure of $X_{\rm aux}$

$$\{\lambda \in \mathcal{C}^1([0,T]; H^{-1/2}(\Gamma) \,:\, \ddot\lambda \in L^1(\Gamma),\, \lambda(0) = \dot\lambda(0) = 0\}.$$

With this we have proved that the bounds (7.46) can be extended for $u = \mathcal{S} * \lambda$ with $\lambda \in \mathcal{W}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$. This proves the mapping property for the operator $\lambda \mapsto \mathcal{K}^t * \lambda$ in the statement, as well as the bound (7.45).

If $\lambda \in \mathcal{W}^1_+(\mathbb{R}; H^{-1/2}(\Gamma))$, then $\partial^{-1}\lambda \in \mathcal{W}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$. Considering then $u := \mathcal{S} * \partial^{-1}\lambda$, and noticing that $\dot u = \mathcal{S} * \lambda$, we have proved (7.42). Using (7.46b), we prove (7.43a). The continuity of $\mathcal{V} * \lambda$ and (7.43b) are a straightforward consequence of the fact that $\mathcal{V} * \lambda = \gamma(\mathcal{S} * \lambda)$. □

Theorem 7.5.2 (Mapping properties of the Galerkin error operator for $\mathcal{V}$). *If $\lambda \in \mathcal{W}^1_+(\mathbb{R}; H^{-1/2}(\Gamma))$, then*

$$\mathcal{E}^h_u * \lambda \in \mathcal{C}^1_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1(\mathbb{R}^d))$$

and for all $t \geq 0$

$$\|(\mathcal{E}^h_u * \lambda)(t)\|_{1,\mathbb{R}^d} \leq C\, H_2(\partial^{-1}\lambda, t \,|\, H^{-1/2}(\Gamma)).$$

If additionally $\lambda \in \mathcal{W}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$, then

$$\mathcal{E}^h_\lambda * \lambda \in \mathcal{C}_+(\mathbb{R}; H^{-1/2}(\Gamma))$$

and for all $t \geq 0$

$$\|(\mathcal{E}^h_\lambda * \lambda)(t)\|_{-1/2,\Gamma} \leq C\, H_2(\lambda, t \,|\, H^{-1/2}(\Gamma)).$$

All the bounds are independent of the choice of the discrete space X_h.

Proof. The argument is very similar to the one of Theorem 7.5.1. Given $\lambda \in \mathcal{C}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$, we define $\varepsilon^h_u := \mathcal{E}^h_u * \lambda$ and $\varepsilon^h_\lambda = [\![\partial_\nu \varepsilon^h_u]\!] = \mathcal{E}^h_\lambda * \lambda$. The results of Section 7.3 (see Propositions 7.3.1 and 7.3.4 and the conclusions at the end of the section) prove that

$$\varepsilon^h_u \in \mathcal{C}^2_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1_+(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$$

and

$$\begin{aligned}
\|\varepsilon^h_u(t)\|_{1,\mathbb{R}^d} &\leq C(\|\lambda(t)\|_{-1/2,\Gamma} + c(t)\, B(\lambda, t)),\\
\|\dot{\varepsilon}^h_u(t)\|_{1,\mathbb{R}^d} &\leq C(\|\dot{\lambda}(t)\|_{-1/2,\Gamma} + B(\lambda, t)),\\
\|\ddot{\varepsilon}^h_u(t)\|_{\mathbb{R}^d\setminus\Gamma} + \|[\![\partial_\nu \varepsilon^h_u]\!](t)\|_{-1/2,\Gamma} &\leq C(\|\lambda(t)\|_{-1/2,\Gamma} + B(\lambda, t)).
\end{aligned}$$

The remainder of the proof of Theorem 7.5.1 can be applied almost verbatim: first we apply a density argument to weaken λ to be in $\mathcal{W}^2_+(\mathbb{R}; H^{-1/2}(\Gamma))$ and then we use a shifting argument to work with $\mathcal{E}^h_\lambda * \lambda = \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}^h_\lambda * \partial^{-1}\lambda)$. □

Theorem 7.5.3 (Mapping properties of the Galerkin solver for $\mathcal{V}$). *If $\beta \in \mathcal{W}^2_+(\mathbb{R}; H^{1/2}(\Gamma))$, then*

$$\mathcal{G}^h_u * \beta \in \mathcal{C}^1_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1(\mathbb{R}^d))$$

and for all $t \geq 0$

$$\|(\mathcal{G}^h_u * \beta)(t)\|_{1,\mathbb{R}^d} \leq C\, H_3(\partial^{-1}\beta, t \,|\, H^{1/2}(\Gamma)).$$

If additionally $\beta \in \mathcal{W}^3_+(\mathbb{R}; H^{1/2}(\Gamma))$, then

$$\mathcal{G}^h_\lambda * \beta \in \mathcal{C}_+(\mathbb{R}; H^{-1/2}(\Gamma))$$

and for all $t \geq 0$

$$\|(\mathcal{G}^h_\lambda * \beta)(t)\|_{-1/2,\Gamma} \leq C\, H_3(\beta, t \,|\, H^{1/2}(\Gamma)).$$

All the bounds are independent of the choice of the discrete space X_h and therefore, the above assertions hold for the case $X_h = H^{-1/2}(\Gamma)$, in which case $\mathcal{G}^h_u = \mathcal{D} * \mathcal{V}^{-1}$ and $\mathcal{G}^h_\lambda = \mathcal{V}^{-1}$.

Proof. It is again very similar to the proofs of Theorems 7.5.1 and 7.5.2. We start with $\beta \in \mathcal{C}^3_+(\mathbb{R}; H^{1/2}(\Gamma))$, define $u^h := \mathcal{S} * \mathcal{G}^h_\lambda = \mathcal{G}^h_u * \lambda$ and note that by the results of Section 7.4,

$$u^h \in \mathcal{C}^2_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1_+(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma))$$

and

$$\begin{aligned}
\|u^h(t)\|_{1,\mathbb{R}^d} &\le C(\|\beta(t)\|_{1/2,\Gamma} + c(t)\, B(\beta, t)),\\
\|\dot u^h(t)\|_{1,\mathbb{R}^d} &\le C(\|\beta(t)\|_{1/2,\Gamma} + \|\dot\beta(t)\|_{1/2,\Gamma} + B(\beta, t) + B(\dot\beta, t)),\\
\|\ddot u^h(t)\|_{\mathbb{R}^d\setminus\Gamma} + \|[\![\partial_\nu u^h]\!](t)\|_{-1/2,\Gamma} &\le C(\|\beta(t)\|_{1/2,\Gamma} + \|\dot\beta(t)\|_{1/2,\Gamma} + B(\dot\beta, t)).
\end{aligned}$$

(These bounds come from the collection of bounds in Propositions 7.4.1 and 7.4.2.) The remainder of the proof follows the usual pattern of a density argument followed by a shifting argument. □

7.6 Exercises

1. (Section 7.1) Prove that if

$$\|v\|_{B_0} \le C_0 \|\nabla v\|_{B_0} \qquad \forall v \in H^1_0(B_0),$$

   then

$$\|v\|_{B_T} \le C_0(1 + T/R)\|\nabla v\|_{B_T} \qquad \forall v \in H^1_0(B_T).$$

2. (Section 7.1) Prove Lemma 7.1.2.

3. (Section 7.2) Prove the bounds given in Proposition 7.2.1.

4. (Section 7.2) Let $\xi \in \mathrm{TD}(H^{1/2}(\partial B_T))$. Show that there exists

$$\mathcal{M} \in \mathrm{TD}(\mathcal{B}(H^{1/2}(\partial B_T), H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma))$$

   such that $w = \mathcal{M} * \xi$ is the unique solution in $\mathrm{TD}(H^1(B_T) \cap H^1_\Delta(B_T \setminus \Gamma))$ of

$$\ddot w = \Delta w \qquad [\![\partial_\nu w]\!] = 0 \qquad \gamma_T w = \xi.$$

   (**Hint.** This can be done using the Laplace transform or writing w in terms of potentials and operators defined on the boundary ∂B_T.)

5. (Section 7.2) Let $\lambda \in \mathrm{TD}(H^{-1/2}(\Gamma))$ be such that $\lambda|_{(0,\infty)} \in \mathcal{C}^2_0([0,\infty); H^{-1/2}(\Gamma))$ and let $u := \mathcal{S} * \lambda$. Show that

$$u \in \mathcal{C}^2([0,\infty); L^2(\mathbb{R}^d)) \cap \mathcal{C}^1([0,\infty); H^1(\mathbb{R}^d)) \cap \mathcal{C}([0,\infty); H^1_\Delta(\mathbb{R}^d \setminus \Gamma)).$$

   Use the initial values and the differential equation to show that this property can be extended through the origin $t = 0$ to

$$u \in \mathcal{C}^2(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma)).$$

6. (Section 7.3) Let $X_h \subset H^{-1/2}(\Gamma)$ be closed and let

$$V := \{v \in H_0^1(B_T) \,:\, \gamma v \in X_h^\circ\}.$$

Show that if $u \in H^1_\Delta(B_T \setminus \Gamma)$, then

$$[\![\partial_\nu u]\!] \in X_h \qquad \Longleftrightarrow \qquad (\nabla u, \nabla v)_{B_T\setminus\Gamma} + (\Delta u, v)_{B_T\setminus\Gamma} = 0 \quad \forall v \in V.$$

7. (Section 7.3) Show that problems (7.21) and (7.22) are equivalent.

8. (Section 7.3) Let ε_T and ε_{T+M} be the solutions of (7.24) for cut-off domains B_T and B_{T+M} respectively, where $M > 0$.

   (a) Use the Waiting Time property (as phrased in Proposition 7.3.2) to show that the function

   $$\underline{\varepsilon}_T : [0,\infty) \to H_0^1(B_{T+M}) \cap H^1_\Delta(B_{T+M} \setminus (\Gamma \cup \partial B_T))$$

   given by

   $$\underline{\varepsilon}_T(t)(\mathbf{x}) := \begin{cases} \varepsilon_T(t)(\mathbf{x}), & \mathbf{x} \in B_T, \\ 0, & \mathbf{x} \in B_{T+M} \setminus B_T, \end{cases}$$

   is a strong solution of the same problem as ε_{T+M} in the time interval $[0, T+\delta]$.

   (b) Use an energy conservation argument (see the final arguments in the proof of Proposition 6.3.2) to show that $\varepsilon_{T+M} = \underline{\varepsilon}_T$ in $[0, T+\delta]$.

9. (Section 7.3) Give all details leading to the proof of (7.30) and (7.31). (**Hint.** The arguments are the same as those used at the end of Section 7.2 to derive regularity and estimates for the single layer potential.)

10. (Section 7.4) Show that problems (7.33) and (7.34) are equivalent and uniquely solvable.

11. (Section 7.4) **Weak solutions are distributional solutions.** Consider the function $u_T^h : [0,\infty) \to H_0^1(B_T)$ defined by (7.36)-(7.37) and assume that β is polynomially bounded.

    (a) Show that we can understand $Eu_T^h \in \mathrm{TD}(H_0^1(B_T))$ and also $Eu_T^h \in \mathrm{TD}(\mathbb{X}_T)$. (**Hint.** Use the last part of Proposition 6.5.1 to study the extension of $\int_0^t u_T^h$.)

    (b) Show that Eu_T^h is the unique solution of the problem

    $$w \in \mathrm{TD}(\mathbb{X}_T) \qquad \ddot{w} = \Delta w, \qquad \gamma w - E\beta \in X_h^\circ, \qquad [\![\partial_\nu w]\!] \in X_h.$$

    (**Hint.** Define $\mathrm{U}(s) := \mathcal{L}\{Eu_T^h\} \in \mathbb{X}_T$ and show that

    $$\gamma \mathrm{U}(s) - \mathcal{L}\{E\beta\} \in X_h^\circ \qquad s^2\langle \mathrm{U}(s), v\rangle_{V'\times V} + (\nabla \mathrm{U}(s), \nabla v)_{B_T} = 0 \qquad \forall v \in V.$$

    This can be done in several different ways, one of which consists of rewriting (7.36a) as distributional equation in V'.)

The slightly surprising result of this exercise consists of the fact that even if the strong equations are satisfied in V' (because there is no regularity to have them in $L^2(B_T)$ and to separate the natural transmission condition), the distributional equation is actually satisfied in $L^2(B_T)$ and the natural transmission condition is satisfied in the sense of distributions. The reader should compare this with the paradox of the moving waves in the exercise list of Chapter 2.

12. (Section 7.5) Show that

$$H_k(\partial^{-1}\psi, t \,|\, X) \leq t \int_0^t \|\psi(\tau)\|_X \mathrm{d}\tau + H_{k-1}(\psi, t \,|\, X).$$

Chapter 8

The double layer potential

This chapter is the double layer counterpart of Chapters 5 and 7. We will start by studying

$$\mathcal{D} * \varphi, \qquad \mathcal{K} * \varphi, \quad \text{and} \quad \mathcal{W} * \varphi.$$

We will next focus on the semidiscrete inversion of the convolution operator $\varphi \mapsto \mathcal{W} * \varphi$.

8.1 Semidiscrete inverse and Galerkin error

The Galerkin solver. We choose a closed subspace $Y_h \subset H^{1/2}(\Gamma)$ and first think of the following Galerkin solver: for causal $\alpha \in \mathrm{TD}(H^{-1/2}(\Gamma))$ find a causal Y_h-valued distribution φ^h such that

$$\langle \mathcal{W} * \varphi^h, \psi^h \rangle_\Gamma = \langle \alpha, \psi^h \rangle_\Gamma \qquad \forall \psi^h \in Y_h \tag{8.1}$$

and then define

$$u^h := \mathcal{D} * \varphi^h. \tag{8.2}$$

Note that (8.1) can be written in our usual shorthand notation

$$\mathcal{W} * \varphi^h - \alpha \in Y_h^\circ := \{\mu \in H^{-1/2}(\Gamma) \,:\, \langle \mu, \psi^h \rangle_\Gamma = 0 \quad \forall \psi^h \in Y_h\}. \tag{8.3}$$

Note also that from the point of view of u^h, we have

$$[\![\gamma u^h]\!] \in Y_h, \qquad [\![\partial_\nu u^h]\!] = 0, \qquad \partial_\nu u^h + \alpha \in Y_h^\circ, \tag{8.4}$$

and that the density is recovered with the jump of the trace $\varphi^h := -[\![\gamma u^h]\!]$. We will write

$$\varphi^h = \mathcal{G}^h_\varphi * \alpha \tag{8.5}$$

and avoid any symbol for the operator $\alpha \mapsto \mathcal{D} * \mathcal{G}^h_\varphi * \alpha$ to avoid confusion with the Galerkin solver of Chapter 7.

The Galerkin error operator. As usual, we think of equations (8.1) as a discretization of the equation from the point of view of an exact solution. We thus start with $\varphi \in \mathrm{TD}(H^{1/2}(\Gamma)$ and look for a causal Y_h-valued distribution φ^h such that

$$\langle \mathcal{W} * (\varphi^h - \varphi), \psi^h \rangle_\Gamma = 0 \qquad \forall \psi^h \in Y_h. \tag{8.6}$$

What we care about is the error operator

$$\mathcal{E}^h_\varphi * \varphi = \varphi^h - \varphi = \mathcal{G}^h_\varphi * \mathcal{W} * \varphi - \varphi \tag{8.7}$$

and its associated potential

$$\varepsilon^h_u := \mathcal{D} * \mathcal{E}^h_\varphi * \varphi = \mathcal{D} * \varphi^h - \mathcal{D} * \varphi. \tag{8.8}$$

Once more, from the point of view of the potential, we can write

$$[\![\gamma \varepsilon^h_u]\!] + \varphi \in Y_h, \qquad [\![\partial_\nu \varepsilon^h_u]\!] = 0, \qquad \partial_\nu \varepsilon^h_u \in Y^\circ_h. \tag{8.9}$$

These equations are to be compared with (8.4). At this time, the reader should be able to recognize how in this case the Galerkin solver is going to be simpler to analyze than the error operator, just by looking at what transmission condition is non-homogeneous.

The corresponding symbols for the continuous problem. Let us first refresh our memory with the continuous operators. For

$$\mathrm{D} := \mathcal{L}\{\mathcal{D}\}, \qquad \mathrm{W} := \mathcal{L}\{\mathcal{W}\} = -\partial_\nu \mathcal{D}, \qquad \mathrm{K} := \mathcal{L}\{\mathcal{K}\} := \{\!\{\gamma \mathcal{D}\}\!\},$$

we have (see Proposition 3.4.1) for all $s \in \mathbb{C}_+$

$$\begin{aligned} \|\mathrm{W}(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} &\le C\frac{|s|^2}{\sigma\underline{\sigma}}, & (8.10)\\ \|\mathrm{D}(s)\|_{H^{1/2}(\Gamma)\to H^{1}(\mathbb{R}^d\setminus\Gamma)} &\le C\frac{|s|^{3/2}}{\sigma\,\underline{\sigma}^{3/2}}, & (8.11)\\ \|\mathrm{K}(s)\|_{H^{1/2}(\Gamma)\to H^{1}(\mathbb{R}^d\setminus\Gamma)} &\le C\frac{|s|^{3/2}}{\sigma\,\underline{\sigma}^{3/2}}. & (8.12) \end{aligned}$$

Since for each value of $s \in \mathbb{C}_+$, $\mathrm{K}(s)$ is the transpose (not the adjoint) for $\mathrm{K}^t(s) : H^{-1/2} \to H^{1/2}(\Gamma)$ and $\mathrm{K}^t(s) = \{\!\{\partial_\nu \mathrm{S}(s)\}\!\}$, the bound (8.12) is also a consequence of the bound we got as early as in Proposition 2.6.2. We finally recall the coercivity estimate of Proposition 3.4.1

$$\mathrm{Re}\left(e^{-\imath \mathrm{Arg}\, s}\, \langle \mathrm{W}(s)\varphi, \overline{\varphi}\rangle_\Gamma\right) \ge C\,\frac{\sigma\underline{\sigma}^2}{|s|}\|\varphi\|^2_{1/2,\Gamma} \qquad \forall \varphi \in H^{1/2}(\Gamma) \quad \forall s \in \mathbb{C}_+. \tag{8.13}$$

Symbols for the semidiscrete operators. We now consider

$$\mathrm{G}_\varphi^h := \mathcal{L}\{\mathcal{G}_\varphi^h\} \quad \text{and} \quad \mathrm{E}_\varphi^h := \mathcal{L}\{\mathcal{E}_\varphi^h\}$$

and note that

$$\mathrm{E}_\varphi^h(s) = \mathrm{G}_\varphi^h(s)\mathrm{W}(s) - \mathrm{I}.$$

It follows from the coercivity estimate (8.13) that

$$\|\mathrm{G}_\varphi^h(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C\frac{|s|}{\sigma\,\underline{\sigma}^2} \qquad \forall s \in \mathbb{C}_+.$$

We will need to study this operator again though, given our interest in the postprocessed operator $\mathrm{D}(s)\mathrm{G}_\varphi^h(s)$.

8.2 Laplace domain analysis

A useful space. The proofs of all the assertions in this paragraph are proposed as exercises. We first consider the space

$$H_h := \{u \in H^1(\mathbb{R}^d \setminus \Gamma) \,:\, [\![\gamma u]\!] \in Y_h\},$$

note that the operator that maps $H_h \ni u \mapsto (\gamma^- u, [\![\gamma u]\!])$ has $H^{1/2}(\Gamma) \times Y_h$ as range, and then use this fact to prove this weak (but equivalent) formulation of the natural boundary conditions that we will be using.

Lemma 8.2.1. *Let $u \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$. Then the conditions*

$$[\![\partial_\nu u]\!] = 0, \qquad \partial_\nu u + \alpha \in Y_h^\circ$$

are equivalent to

$$(\nabla u, \nabla v)_{\mathbb{R}^d\setminus\Gamma} + (\Delta u, v)_{\mathbb{R}^d\setminus\Gamma} = -\langle \alpha, [\![\gamma v]\!]\rangle_\Gamma \quad \forall v \in H_h.$$

Analysis of the Galerkin solver. We will follow the four step program of Sections 5.3 and 5.4. Details are left to the reader. We start with $\alpha \in H^{-1/2}(\Gamma)$ and want to study

$$\varphi^h = \mathrm{G}_\varphi^h(s)\alpha \qquad \text{and} \qquad u^h = \mathrm{D}(s)\mathrm{G}_\varphi^h(s)\alpha = \mathrm{D}(s)\varphi^h$$

noticing that $\varphi^h = -[\![\gamma u^h]\!]$.

- Step # 1 (Transmission problem). $u^h = \mathrm{D}(s)\mathrm{G}_\varphi^h(s)\alpha$ if and only if

$$u^h \in H^1(\mathbb{R}^d \setminus \Gamma), \tag{8.14a}$$
$$\Delta u^h - s^2 u^h = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{8.14b}$$
$$[\![\gamma u^h]\!] \in Y_h, \tag{8.14c}$$
$$[\![\partial_\nu u^h]\!] = 0, \tag{8.14d}$$
$$\partial_\nu u^h + \alpha \in Y_h^\circ. \tag{8.14e}$$

Note that, compared to the transmission problems related to the single layer potential (Propositions 5.3.1 and 5.4.1), there is one more transmission condition. In fact, in those problems, the condition $[\![\gamma\cdot]\!] = 0$ was implicit to the fact that solutions were elements of $H^1(\mathbb{R}^d)$.

- Step #2 (Variational formulation) Problem (8.14) is equivalent to

$$\left[\begin{array}{l} u^h \in H_h, \\ a_{s,\mathbb{R}^d\setminus\Gamma}(u^h, v^h) = -\langle \alpha, \gamma v^h\rangle_\Gamma \qquad \forall v^h \in H_h. \end{array}\right. \tag{8.15}$$

- Step # 3 (Energy estimate) The solution of (8.15) satisfies

$$\|\!| u^h \|\!|_{|s|,\mathbb{R}^d\setminus\Gamma} \le C\,\frac{|s|}{\sigma\,\underline{\sigma}}\|\alpha\|_{-1/2,\Gamma}$$

and therefore

$$\|\mathrm{D}(s)\mathrm{G}^h_\varphi(s)\|_{H^{-1/2}(\Gamma)\to H^1(\mathbb{R}^d\setminus\Gamma)} \le C\frac{|s|}{\sigma\underline{\sigma}^2} \qquad \forall s \in \mathbb{C}_+. \tag{8.16}$$

- Step # 4 (Boundary wrap-up). Taking the jump of the trace in (8.16), we obtain

$$\|\mathrm{G}^h_\varphi(s)\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C\frac{|s|}{\sigma\,\underline{\sigma}^2} \qquad \forall s \in \mathbb{C}_+, \tag{8.17}$$

which was already known using a direct coercivity estimate.

Analysis of the Galerkin error operator. We now focus on the operators

$$\mathrm{E}^h_\varphi(s) = \mathrm{G}^h_\varphi(s)\mathrm{W}(s) - \mathrm{I} \qquad \text{and} \qquad \mathrm{D}(s)\mathrm{E}^h_\varphi(s).$$

The four step process goes in parallel to the one above, with the only difference that we will have to come up with a proper lifting of the essential transmission condition (as we did in Proposition 5.3.3 for the Single Layer potential) at the time of obtaining the energy estimate. This being an error analysis, we start with $\varphi \in H^{1/2}(\Gamma)$ and proceed from here.

- Step # 1 (Transmission problem). $\varepsilon^h_u = \mathrm{D}(s)\mathrm{E}^h_\varphi(s)\varphi$ if and only if

$$\varepsilon^h_u \in H^1(\mathbb{R}^d \setminus \Gamma), \tag{8.18a}$$
$$\Delta\varepsilon^h_u - s^2\varepsilon^h_u = 0 \qquad \text{in } \mathbb{R}^d\setminus\Gamma, \tag{8.18b}$$
$$[\![\gamma\varepsilon^h_u]\!] + \varphi \in Y_h, \tag{8.18c}$$
$$[\![\partial_\nu\varepsilon^h_u]\!] = 0, \tag{8.18d}$$
$$\partial_\nu\varepsilon^h_u \in Y_h^\circ. \tag{8.18e}$$

- Step #2 (Variational formulation) Problem (8.18) is equivalent to

$$\left[\begin{array}{ll} \varepsilon^h_u \in H^1(\mathbb{R}^d\setminus\Gamma) & [\![\gamma\varepsilon^h_u]\!] + \varphi \in Y_h, \\ a_{s,\mathbb{R}^d\setminus\Gamma}(\varepsilon^h_u, v^h) = 0 & \forall v^h \in H_h. \end{array}\right. \tag{8.19}$$

- Step # 3 (Energy estimate) Using the lifting of Proposition 2.5.1 in Ω_-, extended by zero to the exterior domain, we can build

$$u_\varphi(s) \in H^1(\mathbb{R}^d \setminus \Gamma), \qquad [\![u_\varphi(s)]\!] = \varphi, \qquad \| u_\varphi(s) \|_{|s|,\mathbb{R}^d\setminus\Gamma} \le C\, \frac{|s|^{1/2}}{\underline{\sigma}^{1/2}} \|\varphi\|_{1/2,\Gamma}.$$

  Then, $\varepsilon_u^h + u_\varphi(s) \in H_h$ can be used as a test in (8.19) and therefore the solution of (8.19) satisfies

$$\| \varepsilon_u^h \|_{|s|,\mathbb{R}^d\setminus\Gamma} \le \| \varepsilon_u^h + u_\varphi(s) \|_{|s|,\mathbb{R}^d\setminus\Gamma} + \| u_\varphi(s) \|_{|s|,\mathbb{R}^d\setminus\Gamma} \le \left(\frac{|s|}{\sigma} + 1 \right) \| u_\varphi(s) \|_{|s|,\mathbb{R}^d\setminus\Gamma}.$$

  From these inequalities we easily derive the bound

$$\|\mathrm{D}(s)\mathrm{E}_\varphi^h(s)\|_{H^{1/2}(\Gamma)\to H^1(\mathbb{R}^d\setminus\Gamma)} \le C\frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}} \qquad \forall s \in \mathbb{C}_+. \tag{8.20}$$

- Step # 4 (Boundary wrap-up). Taking the jump of the trace in (8.20), we finally prove

$$\|\mathrm{E}_\varphi^h(s)\|_{H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)} \le C\frac{|s|^{3/2}}{\sigma\underline{\sigma}^{3/2}} \qquad \forall s \in \mathbb{C}_+. \tag{8.21}$$

8.3 The cut-off process

Geometry. The set up is the same as in Chapter 7. We write

$$\overline{\Omega^-} \subset B_0 := B(\mathbf{0}; R), \qquad B_T := B(\mathbf{0}; R+T), \qquad \delta := \mathrm{dist}\,(\partial B_0, \Gamma)$$

and consider the operators on ∂B_T

$$\gamma_T : H^1(B_T \setminus \overline{\Omega^-}) \to H^{1/2}(\partial B_T) \quad \text{and} \quad \partial_T^\nu : H^1_\Delta(B_T \setminus \overline{\Omega^-}) \to H^{-1/2}(\partial B_T).$$

Inequalities. We now collect an extended version of the inequality toolbox of Section 7.1. Two new spaces will be relevant. The first one contains a Dirichlet boundary condition on ∂B_T but allows jumps across Γ:

$$H^1_{\partial B_T}(B_T \setminus \Gamma) := \{ u \in H^1(B_T \setminus \Gamma) \,:\, \gamma_T u = 0 \}.$$

It is clear that $\|\nabla \ \cdot\ \|_{B_T\setminus\Gamma}$ does not define a norm in $H^1_{\partial B_T}(B_T \setminus \Gamma)$, but the simple correction

$$\|\nabla u\|^2_{B_T\setminus\Gamma} + |\jmath(u)|^2, \quad \text{where} \quad \jmath(u) := \frac{1}{|\Omega^-|^{1/2}} \int_{\Omega^-} u, \tag{8.22}$$

does. The second space includes the Laplacian

$$\mathbb{Y}_T := \{ u \in H^1_{\partial B_T}(B_T \setminus \Gamma) \,:\, \Delta u \in L^2(B_T \setminus \Gamma) \} = \{ u \in H^1_\Delta(B_T \setminus \Gamma) \,:\, \gamma_T u = 0 \}, \tag{8.23}$$

and is endowed with the seminorm

$$|u|^2_{\mathbb{Y}_T} := \|\nabla u\|^2_{B_T\setminus\Gamma} + \|\Delta u\|^2_{B_T\setminus\Gamma}.$$

(a) In addition to the Poincaré-Friedrichs inequality

$$\|v\|_{B_T} \le C_T \|\nabla v\|_{B_T} \quad \forall v \in H_0^1(B_T), \qquad C_T = C_0(1 + T/R), \tag{8.24}$$

we will need a scale-dependent generalized Poincaré inequality

$$\|u\|_{B_T} \le \widehat{C}_T \Big(\|\nabla u\|^2_{B_T \setminus \Gamma} + |\jmath(u)|^2 \Big)^{1/2} \qquad \forall u \in H^1_{\partial B_T}(B_T \setminus \Gamma). \tag{8.25}$$

We could *conjecture* that we can bound $\widehat{C}_T \le C_1 + C_2 T$. This will be not relevant in the sequel, since the constant $\widehat{C}_T$ will be eliminated from all expressions in the final version of the estimates.

(b) Boundedness of the trace operators. For simplicity will use a joint constant for the jump and average of the trace on Γ:

$$\|[\![\gamma u]\!]\|_{1/2,\Gamma} + \|\{\!\!\{\gamma u\}\!\!\}\|_{1/2,\Gamma} \le C_\gamma \|u\|_{1,B_0 \setminus \Gamma} \qquad \forall u \in H^1(B_0 \setminus \Gamma). \tag{8.26}$$

(c) Boundedness of the normal derivative operator:

$$\|\partial_\nu^\pm u\|_{-1/2,\Gamma} \le C_\nu \Big(\|\nabla u\|^2_{B_0 \cap \Omega_\pm} + \|\Delta u\|^2_{B_0 \cap \Omega_\pm} \Big)^{1/2} \qquad \forall u \in H^1(B_0 \setminus \Gamma). \tag{8.27}$$

(d) A one sided lifting of the trace will be used as a lifting of the jump of the trace across Γ. We consider an operator $[\![\gamma]\!]^\dagger : H^{1/2}(\Gamma) \to H^1_{\partial B_0}(B_0 \setminus \Gamma)$ and a constant $C_\gamma^\dagger > 0$ such that

$$[\![[\![\gamma]\!]^\dagger \alpha]\!] = \alpha, \quad \|[\![\gamma]\!]^\dagger \alpha\|_{1,B_0\setminus\Gamma} = \|[\![\gamma]\!]^\dagger \alpha\|_{1,\Omega_-} \le C_\gamma^\dagger \|\alpha\|_{1/2,\Gamma} \quad \forall \alpha \in H^{1/2}(\Gamma). \tag{8.28}$$

8.4 Potentials and operators

We start with the operators

$$\mathcal{D} * \varphi, \qquad \mathcal{W} * \varphi = -\partial_\nu(\mathcal{D} * \varphi), \quad \text{and} \quad \mathcal{K} * \varphi = \{\!\!\{\gamma(\mathcal{D} * \varphi)\}\!\!\}.$$

In terms of the potential, the cut-off problem is the search for $u_T : [0, \infty) \to \mathbb{Y}_T$ (this includes a homogeneous Dirichlet boundary condition on ∂B_T) such that

$$\ddot{u}_T(t) = \Delta u_T(t) \qquad \forall t \ge 0, \tag{8.29a}$$
$$[\![\gamma u_T]\!](t) + \varphi(t) = 0 \qquad \forall t \ge 0, \tag{8.29b}$$
$$[\![\partial_\nu u_T]\!](t) = 0 \qquad \forall t \ge 0, \tag{8.29c}$$
$$u_T(0) = \dot{u}_T(0) = 0. \tag{8.29d}$$

(Compare with (7.5), the corresponding cut-off problem for the single layer potential.) The process will mimick that of the Galerkin solver for the single layer operator, using first a weak solution to get some bounds and then a strong solution to obtain what is left.

The dynamical system triplet. The spaces and norms are the same as those of Section 7.2, namely

$$\begin{aligned} H &:= L^2(B_T), \\ V &:= H_0^1(B_T), \\ D(A) &:= H_0^1(B_T) \cap H^2(B_T) = H_0^1(B_T) \cap H^1_\Delta(B_T) \\ &= \{u \in V \,:\, \Delta u \in L^2(B_T \setminus \Gamma), \quad [\![\partial_\nu u]\!] = 0\}, \end{aligned}$$

with

$$\|u\|_H := \|u\|_{B_T}, \qquad [u,v] := (\nabla u, \nabla v)_{B_T}.$$

The lifting. We consider the operator $L : H^{1/2}(\Gamma) \to \mathbb{Y}_T$ given by the solution of

$$\begin{aligned} & u_0 \in H^1_{\partial B_T}(B_T \setminus \Gamma), \\ & -\Delta u_0 + u_0 = 0 \qquad \text{in } B_T \setminus \Gamma, \\ & [\![\gamma u_0]\!] = -\varphi, \\ & [\![\partial_\nu u_0]\!] = 0, \end{aligned}$$

that is,

$$\left[\begin{array}{l} u_0 \in H^1_{\partial B_T}(B_T \setminus \Gamma), \qquad [\![\gamma u_0]\!] = -\varphi, \\ (\nabla u_0, \nabla v)_{B_T \setminus \Gamma} + (u_0, v)_{B_T} = 0 \qquad \forall v \in H_0^1(B_T). \end{array}\right.$$

Testing then with $v = u_0 + [\![\gamma]\!]^\dagger \varphi$, it is easy to prove that

$$\|L\|_{H^{1/2}(\Gamma) \to Z} \le C_\gamma^\dagger \qquad Z \in \{L^2(B_T), H^1(B_T \setminus \Gamma), \mathbb{Y}_T\}. \tag{8.30}$$

Proposition 8.4.1 (Weak solutions in bounded domain)**.** *Let $\varphi \in \mathcal{C}_0^2([0,\infty); H^{1/2}(\Gamma))$. Then, there exists (a unique)*

$$u_T : [0,\infty) \to H^1_{\partial B_T}(B_T \setminus \Gamma)$$

such that

$$\langle \ddot{u}_T(t), v\rangle_{V' \times V} + (\nabla u_T(t), \nabla v)_{B_T \setminus \Gamma} = 0 \qquad \forall v \in V \qquad \forall t \ge 0, \tag{8.31a}$$

$$[\![\gamma u_T]\!](t) + \varphi(t) = 0 \qquad \forall t \ge 0, \tag{8.31b}$$

$$u_T(0) = \dot{u}_T(0) = 0, \tag{8.31c}$$

with regularity

$$u_T \in \mathcal{C}^2([0,\infty); V') \cap \mathcal{C}^1([0,\infty); L^2(B_T)) \cap \mathcal{C}([0,\infty); H^1_{\partial B_T}(B_T \setminus \Gamma)). \tag{8.32}$$

Moreover, for all $t \ge 0$,

$$\begin{aligned} \|u_T(t)\|_{B_T} &\le C_\gamma^\dagger \big(\|\varphi(t)\|_{1/2,\Gamma} + C_T B(\varphi, t)\big), \\ \|\nabla u_T(t)\|_{B_T \setminus \Gamma} &\le C_\gamma^\dagger \big(\|\varphi(t)\|_{1/2,\Gamma} + B(\varphi, t)\big), \\ \|\dot{u}_T(t)\|_{B_T} &\le C_\gamma^\dagger \big(\|\dot{\varphi}(t)\|_{1/2,\Gamma} + B(\varphi, t)\big), \\ \|\{\!\{\gamma u_T\}\!\}(t)\|_{1/2,\Gamma} &\le C_\gamma C_\gamma^\dagger \big(\|\varphi(t)\|_{1/2,\Gamma} + c(T)\, B(\varphi, t)\big) \end{aligned}$$

where

$$c(T) := \sqrt{1 + C_T^2} \quad \textit{and} \quad B(\varphi, t) := \int_0^t \|\varphi(\tau) - \ddot{\varphi}(\tau)\|_{1/2,\Gamma} \mathrm{d}\tau.$$

Proof. The entire proof is based on showing that if we define $u_0(t) := L\varphi(t)$, $f(t) := u_0(t) - \ddot{u}_0(t)$, and $v_0 : [0, \infty) \to H_0^1(B_T) = V$ is the solution to the problem

$$\begin{aligned}
&\langle \ddot{v}_0(t), v \rangle_{V' \times V} + (\nabla v_0(t), \nabla v)_{B_T} = (f(t), v)_{B_T} \quad \forall v \in V \qquad \forall t \geq 0, \\
&v_0(0) = \dot{v}_0(0) = 0,
\end{aligned}$$

then $u_T := u_0 + v_0$ is the unique solution of (7.36). It is clear that $u_0 \in \mathcal{C}_0^2([0, \infty); \mathbb{Y}_T)$, while (see Proposition 6.5.1)

$$v_0 \in \mathcal{C}^2([0, \infty); V') \cap \mathcal{C}^1([0, \infty); L^2(B_T)) \cap \mathcal{C}([0, \infty); H_0^1(B_T)).$$

The bounds follows then from (8.30) (to estimate u_0) and Proposition 6.4.2 (to estimate v_0). To bound $\{\!\{\gamma u_T\}\!\}(t)$, use a bound for the $H^1(B_T)$ norm and (8.26). □

Proposition 8.4.2 (Strong solutions in bounded domain). *Let $\varphi \in \mathcal{C}_0^3([0, \infty); H^{1/2}(\Gamma))$. Then, there exists (a unique)*

$$u_T : [0, \infty) \to \mathbb{Y}_T,$$

satisfying

$$\begin{aligned}
\ddot{u}_T(t) &= \Delta u_T(t) \qquad \forall t \geq 0, && (8.34a)\\
[\![\gamma u_T(t)]\!] + \varphi(t) &= 0 \qquad \forall t \geq 0, && (8.34b)\\
[\![\partial_\nu u_T]\!](t) &= 0 \qquad \forall t \geq 0, && (8.34c)\\
u_T(0) = \dot{u}_T(0) &= 0, && (8.34d)
\end{aligned}$$

with regularity

$$u_T \in \mathcal{C}^2([0, \infty); L^2(B_T)) \cap \mathcal{C}^1([0, \infty); H^1_{\partial B_T}(B_T \setminus \Gamma)) \cap \mathcal{C}([0, \infty); \mathbb{Y}_T). \tag{8.35}$$

Moreover, u_T is also the solution of (8.31) *and*

$$\begin{aligned}
\|\Delta u_T(t)\|_{B_T \setminus \Gamma} &\leq C_\gamma^\dagger (\|\varphi(t)\|_{1/2,\Gamma} + 2B(\dot{\varphi}, t)), \\
\|\nabla \dot{u}_T(t)\|_{B_T \setminus \Gamma} &\leq C_\gamma^\dagger (\|\dot{\varphi}(t)\|_{1/2,\Gamma} + B(\dot{\varphi}, t)), \\
\|\partial_\nu u_T(t)\|_{-1/2,\Gamma} &\leq C_\nu C_\gamma^\dagger (2\|\varphi(t)\|_{1/2,\Gamma} + B(\varphi, t) + 2B(\dot{\varphi}, t)).
\end{aligned}$$

Proof. We now follow the steps of the proof of Proposition 7.4.2. To do that, we define $u_0(t) := L\varphi(t)$, $f(t) := u_0(t) - \ddot{u}_0(t)$, as in the proof of Proposition 8.4.1. We then consider $v_0 : [0, \infty) \to D(A)$ as the solution of

$$\ddot{v}_0(t) = \Delta v_0(t) + f(t) \quad \forall t \geq 0, \qquad v_0(t) = \dot{v}_0(t) = 0.$$

Note that $u_0 \in \mathcal{C}_0^3([0, \infty); \mathbb{Y}_T)$ and

$$v_0 \in \mathcal{C}^2([0, \infty); L^2(B_T)) \cap \mathcal{C}^1([0, \infty); H_0^1(B_T)) \cap \mathcal{C}([0, \infty); D(A)).$$

It is then easy to show that $u_T := u_0 + v_0$ satisfies (8.34) and (8.35). It is also easy to show that a solution of (8.34) is a solution of (8.31).

To bound the Laplacian of u_T we first use (8.30) to obtain

$$\|\Delta u_0(t)\|_{B_T\setminus\Gamma} \le C_\gamma^\dagger \|\varphi(t)\|_{1/2,\Gamma}. \tag{8.36}$$

At the same time, by Proposition 6.4.3 and (8.30), it follows that

$$\|\Delta v_0(t)\|_{B_T} \le 2\int_0^t \|\dot f(\tau)\|_{B_T} \mathrm{d}\tau \le 2C_\gamma^\dagger B(\dot\varphi, t). \tag{8.37}$$

The bound for $\|\Delta u_T(t)\|_{B_T\setminus\Gamma}$ is just the collection of (8.36) and (8.37). The bound for $\|\partial_\nu u_T(t)\|_{-1/2,\Gamma}$ follows now from (8.27), the bound for $\|\Delta u_T(t)\|_{B_T\setminus\Gamma}$ and the estimate of $\|\nabla u_T(t)\|_{B_T\setminus\Gamma}$ in Proposition 8.4.1. □

Proposition 8.4.3 (Waiting Time on ∂B_T)**.** *Let $u_T : [0,\infty) \to \mathbb{Y}_T$ be the solution of* (8.34) *for $\varphi \in \mathcal{C}_0^4([0,\infty); H^{1/2}(\Gamma))$. Then*

$$\partial_T^\nu u_T(t) = 0 \qquad 0 \le t \le T + \delta.$$

Proof. As usual in this kind of results, assume that φ is polynomially bounded as $t \to \infty$. The plan of the proof is similar to that of Proposition 7.2.2. Consider first

$$w := Eu_T - (\mathcal{D} * E\varphi)_{B_T} \in \mathrm{TD}(H^1_\Delta(B_T \setminus \Gamma)), \qquad \xi := -\gamma_T(\mathcal{D} * \varphi) \in \mathrm{TD}(H^{1/2}(\partial B_T)).$$

Prove next that

$$w \in \mathrm{TD}(H^1_\Delta(B_T \setminus \Gamma)), \qquad \ddot w = \Delta w, \qquad [\![\gamma w]\!] = 0, \qquad [\![\partial_\nu w]\!] = 0, \qquad \gamma_T w = \xi,$$

and that this problem is uniquely solvable. Use the finite speed of propagation of the double layer potential (Proposition 7.1.1) to show that $\operatorname{supp}\xi \subset [T+\delta,\infty)$ and therefore

$$\operatorname{supp} w \subset [T+\delta,\infty).$$

Use again Proposition 7.1.1 to show that $\operatorname{supp}\partial_T^\nu E u_T \subset [T+\delta,\infty)$ and finish the proof. □

Proposition 8.4.4 (Extension to free space)**.** *Let $\varphi \in \mathcal{C}_0^3([0,\infty); H^{1/2}(\Gamma))$ and u_T be the solution of* (8.34)*. Consider the extension*

$$\underline{u}_T : [0,\infty) \to H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))$$

given by

$$\underline{u}_T(t)|_{B_T} = u_T(t) \qquad \textit{and} \qquad \underline{u}_T(t)|_{\mathbb{R}^d\setminus B_T} = 0.$$

Then $\underline{u}_T : [0, T+\delta] \to H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ is continuous and

$$\underline{u}_T(t) = (\mathcal{D} * E\varphi)(t) \qquad 0 \le t < T + \delta.$$

Proof. The proof is a slight variant of the proof of Proposition 7.2.3. We first prove that

$$E\underline{u}_T = \mathcal{D} * E\varphi + \mathcal{S}_{\partial B_T} * E\partial_T^\nu u_T$$

and then use Proposition 8.4.3 to show that $\operatorname{supp}(E\underline{u}_T - \mathcal{D} * E\varphi) \subset [T+\delta, \infty)$. The fact that $\underline{u}_T(t) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ for $t \le T + \delta$ is a direct consequence of Proposition 7.2.3. □

Theorem 8.4.5 (Mapping properties for the double layer potential)**.** *If $\varphi \in \mathcal{W}^2_+(\mathbb{R}; H^{1/2}(\Gamma))$, then*

$$\mathcal{D} * \varphi \in \mathcal{C}^1_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1(\mathbb{R}^d \setminus \Gamma))$$

and therefore

$$\mathcal{K} * \varphi \in \mathcal{C}_+(\mathbb{R}; H^{1/2}(\Gamma)).$$

Moreover, for all $t \ge 0$

$$\begin{aligned}
\|(\mathcal{D} * \varphi)(t)\|_{1,\mathbb{R}^d\setminus} &\le C\, H_3(\partial^{-1}\varphi, t \,|\, H^{1/2}(\Gamma)),\\
\|(\mathcal{K} * \varphi)(t)\|_{1/2,\Gamma} &\le C\, H_3(\partial^{-1}\varphi, t \,|\, H^{1/2}(\Gamma)).
\end{aligned}$$

If additionally $\varphi \in \mathcal{W}^3_+(\mathbb{R}; H^{1/2}(\Gamma))$, then

$$\mathcal{W} * \varphi \in \mathcal{C}_+(\mathbb{R}; H^{-1/2}(\Gamma)).$$

and for all $t \ge 0$

$$\|(\mathcal{W} * \varphi)(t)\|_{-1/2,\Gamma} \le C\, H_3(\varphi, t \,|\, H^{1/2}(\Gamma)).$$

Proof. For causal $\varphi \in \mathcal{C}^3(\mathbb{R}; H^{-1/2}(\Gamma))$, we let $u := \mathcal{D} * \varphi$. From the results above (Propositions 8.4.1, 8.4.2, and 8.4.4) it follows that

$$u \in \mathcal{C}^2_+(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1_+(\mathbb{R}; H^1(\mathbb{R}^d \setminus \Gamma)) \cap \mathcal{C}_+(\mathbb{R}; H^1_\Delta(\mathbb{R}^d \setminus \Gamma)).$$

Also, taking $T = t$ in Propositions 8.4.1 and 8.4.2, it follows that

$$\begin{aligned}
\|u(t)\|_{1,\mathbb{R}^d\setminus\Gamma} &\le C(\|\varphi(t)\|_{-1/2,\Gamma} + c(t)B(\varphi, t)),\\
\|\dot{u}(t)\|_{1,\mathbb{R}^d\setminus\Gamma} &\le C(\|\dot{\varphi}(t)\|_{-1/2,\Gamma} + B(\varphi, t) + B(\dot{\varphi}, t)),\\
\|\ddot{u}(t)\|_{\mathbb{R}^d\setminus\Gamma} &\le C(\|\varphi(t)\|_{-1/2,\Gamma} + B(\dot{\varphi}, t)),\\
\|(\partial_\nu u)(t)\|_{-1/2,\Gamma} &\le C(\|\dot{\varphi}(t)\|_{-1/2,\Gamma} + B(\varphi, t) + B(\dot{\varphi}, t)).
\end{aligned}$$

The remainder of the proof follows, line by line, the proof of Theorem 7.5.1. □

8.5 Galerkin solver for the hypersingular operator

Given a causal $H^{-1/2}(\Gamma)$-valued distribution α, we deal here with the discretization of the equation $\mathcal{W} * \varphi = \alpha$. This is carried out on a discrete space $Y_h \subset H^{1/2}(\Gamma)$. We will assume that

$$\mathbb{P}_0(\Gamma) \subset Y_h, \tag{8.38}$$

although, more properly speaking, what we need is $\gamma\mathbb{P}_0(\Omega_-) \subset Y_h$. We then look for a causal $H^{1/2}(\Gamma)$-valued distribution φ^h such that

$$\varphi^h \in Y_h, \qquad \mathcal{W} * \varphi^h - \alpha \in Y_h^\circ$$

and consider the associated potential

$$u^h = \mathcal{D} * \varphi^h.$$

In terms of the potential u^h, we are looking for

$$\begin{aligned} u^h &\in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)), \\ \ddot{u}^h &= \Delta u^h, \\ [\![\gamma u^h]\!] &\in Y_h, \\ [\![\partial_\nu u^h]\!] &= 0, \\ \partial_\nu u^h + \alpha &\in Y_h^\circ. \end{aligned}$$

Cut-off at an external (eventually moving) boundary ∂B_T will be done as usual, leading to a dynamical system, which, as a novelty, will contain a rigid motion.

Dynamical system. We start with

$$H := L^2(B_T) \qquad \|u\|_H := \|u\|_{B_T}, \qquad M := \mathrm{span}\{\chi_{\Omega_-}\},$$

so that the projection onto M is

$$Pu = \Big(\frac{1}{|\Omega_-|}\int_{\Omega_-} u\Big)\chi_{\Omega_-} = \Big(\frac{1}{|\Omega_-|^{1/2}}\int_{\Omega_-} u\Big)\frac{1}{|\Omega_-|^{1/2}}\chi_{\Omega_-}$$

and (see (8.22))

$$\|Pu\|_{B_T} = \|Pu\|_{\Omega_-} = |\jmath(u)|.$$

We next consider

$$V := \{u \in H^1_{\partial B_T}(B_T \setminus \Gamma) \,:\, [\![\gamma u]\!] \in Y_h\} \qquad [u, v] := (\nabla u, \nabla v)_{B_T \setminus \Gamma},$$

as well as

$$D(A) := \{u \in V \,:\, \Delta u \in L^2(B_T \setminus \Gamma) \,:\, [\![\partial_\nu u]\!] = 0, \quad \partial_\nu u \in Y_h^\circ\},$$

with the operator A being the distributional Laplacian in $B_T \setminus \Gamma$. The condition (8.38) is used to show that $M \subset V$. Let us now check all the hypotheses:

- Since $\mathcal{D}(B_T \setminus \Gamma) \subset V \subset H^1(B_T \setminus \Gamma) \subset L^2(B_T)$, the compactness and density of the injection $V \subset H$ is straightforward to prove.
- The constant for the inequality between norms in H and V is given by the constant of the generalized Poincaré inequality (8.25).

- To prove the associated Green's formula, we just need to use Lemma 8.2.1.
- Finally, given $f \in L^2(B_T)$ we can solve

$$\left[\begin{array}{l} u \in V, \\ (\nabla u, \nabla v)_{B_T\setminus\Gamma} + (u, v)_{B_T} = (f, v)_{B_T} \qquad \forall v \in V. \end{array}\right.$$

Then, by Lemma 8.2.1, $u \in D(A)$ and $-\Delta u + u = f$.

Lifting. Consider the operator $L : H^{1/2}(\Gamma) \to \mathbb{Y}_T$ that associated $u_0 = L\alpha$ by solving

$$\left[\begin{array}{l} u_0 \in V, \\ (\nabla u_0, \nabla v)_{B_T\setminus\Gamma} + (u_0, v)_{B_T} = -\langle \alpha, [\![\gamma v]\!]\rangle_\Gamma \qquad \forall \alpha \in V, \end{array}\right. \tag{8.39}$$

or equivalently

$$\begin{aligned} u_0 \in H^1_{\partial B_T}(B_T \setminus \Gamma), & \\ -\Delta u_0 + u_0 = 0 & \qquad \text{in } B_T \setminus \Gamma, \\ [\![\gamma u_0]\!] \in Y_h, & \\ [\![\partial_\nu u_0]\!] = 0, & \\ \partial_\nu u_0 + \alpha \in Y_h^\circ. & \end{aligned}$$

It is then easy to see that

$$\|L\alpha\|_{1,B_T} = |L\alpha|_{\mathbb{Y}_T} \le C_\gamma \|\alpha\|_{-1/2,\Gamma}. \tag{8.40}$$

Testing (8.39) with $v = |\Omega_-|^{-1/2}\chi_{\Omega_-}$, it follows that

$$PL\alpha = -\Big(\frac{1}{|\Omega_-|}\langle \alpha, 1\rangle_\Gamma\Big)\chi_{\Omega_-}. \tag{8.41}$$

Proposition 8.5.1. *Let $\alpha \in \mathcal{C}^2_0([0,\infty); H^{-1/2}(\Gamma))$. Then there exists a unique*

$$u^h_T : [0,\infty) \to H^1_{\partial B_T}(B_T \setminus \Gamma),$$

$$\ddot{u}^h_T(t) = \Delta u^h_T(t) \qquad \forall t \ge 0, \tag{8.42a}$$

$$[\![\gamma u^h_T]\!](t) \in Y_h \qquad \forall t \ge 0, \tag{8.42b}$$

$$[\![\partial_\nu u^h_T]\!](t) = 0 \qquad \forall t \ge 0, \tag{8.42c}$$

$$\partial_\nu u^h_T(t) + \alpha(t) \in Y_h^\circ, \tag{8.42d}$$

$$u^h_T(0) = \dot{u}^h_T(0) = 0, \tag{8.42e}$$

and

$$u^h_T \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H^1(B_T \setminus \Gamma)) \cap \mathcal{C}([0,\infty); \mathbb{Y}_T)). \tag{8.43}$$

Moreover, for all $t \geq 0$,

$$\begin{aligned}
\|u_T^h(t)\|_{1,B_T\setminus\Gamma} &\leq |\Omega_-|^{-1/2}|\rho(t)| + C_\gamma(\|\alpha(t)\|_{-1/2,\Gamma} + \widehat{c}(T)B(\alpha,t)),\\
\|\dot{u}_T^h(t)\|_{1,B_T\setminus\Gamma} &\leq |\Omega_-|^{-1/2}|\dot{\rho}(t)| + C_\gamma(\|\dot{\alpha}(t)\|_{-1/2,\Gamma} + \sqrt{2}\,B(\alpha,t)),\\
\|\Delta u_T^h(t)\|_{B_T\setminus\Gamma} &\leq C_\gamma(\|\alpha(t)\|_{-1/2,\Gamma} + B(\alpha,t)),
\end{aligned}$$

where

$$\widehat{c}(T) := \sqrt{1+\widehat{C}_T^2}, \qquad B(\alpha,t) := \int_0^t \|\alpha(\tau) - \ddot{\alpha}(\tau)\|_{-1/2,\Gamma}\mathrm{d}\tau,$$

and

$$\rho(t) := \int_0^t (t-\tau)\langle\alpha(t),1\rangle_\Gamma \mathrm{d}\tau.$$

Proof. The proof of this result requires taking care of the rigid motion that is excited in the system. This will be done counting on the results in Section 6.6. We define

$$u_0(t) := L\alpha(t), \qquad f(t) := u_0(t) - \ddot{u}_0(t) = L(\alpha(t) - \ddot{\alpha}(t)),$$

consider the rigid motion (see Proposition 6.6.1 and (8.41))

$$\begin{aligned}
m_\alpha(t) &:= Pu_0(t) + m_f(t) = Pu_0(t) + \int_0^t (t-\tau)Pf(\tau)\mathrm{d}\tau\\
&= PL\alpha(t) + \int_0^t (t-\tau)PL\alpha(\tau)\mathrm{d}\tau - \int_0^t (t-\tau)PL\ddot{\alpha}(\tau)\mathrm{d}\tau\\
&= \int_0^t (t-\tau)PL\alpha(\tau)\mathrm{d}\tau = -\frac{1}{|\Omega_-|}\Big(\int_0^t (t-\tau)\langle\alpha(\tau),1\rangle_\Gamma\mathrm{d}\tau\Big)\chi_{\Omega_-} = -\frac{\rho(t)}{|\Omega_-|}\chi_{\Omega_-},
\end{aligned}$$

and then consider the solution (see Proposition 6.4.2) $v_0 : [0,\infty) \to D(A)$ of the evolution equation

$$\ddot{v}_0(t) = \Delta v_0(t) + f(t) - Pf(t) \qquad t \geq 0, \qquad v_0(0) = \dot{v}_0(0) = 0.$$

The solution of (8.42) is easily recomposed as the sum

$$u_T^h = (u_0 - Pu_0) + m_\alpha + v_0 = u_0 + m_f + v_0.$$

Note that

$$u_0 - Pu_0 = (I-P)L\alpha \in \mathcal{C}_0^2([0,\infty);\mathbb{Y}_T), \qquad m_\alpha \in \mathcal{C}_0^4([0,\infty);M),$$

and

$$v_0 \in \mathcal{C}^2([0,\infty);H) \cap \mathcal{C}^1([0,\infty);V) \cap \mathcal{C}([0,\infty);D(A)),$$

which proves (8.43). The end of the proof is proposed as an exercise. □

Identification of u_T^h as the Galerkin solver. The next sequence of steps does not vary from what we have repeatedly done in previous examples:

- By comparison with $\mathcal{D} * [\![\gamma u_T^h]\!]|_{B_T}$, we show that

$$\partial_\nu^T u_T^h(t) = 0 \qquad 0 \le t \le T + \delta.$$

- With a similar argument, we show that we can define

$$\varphi^h(T) := -[\![\gamma u_T^h]\!](T),$$

  that $\varphi^h \in \mathcal{C}^1([0,\infty); H^{1/2}(\Gamma))$ and

$$\varphi^h(t) = -[\![\gamma u_T^h]\!](t) \qquad t \le T + \delta.$$

- Finally, we consider the function $\underline{u}_T^h$ (extension by zero to $\mathbb{R}^d \setminus B_T$) and show that

$$\underline{u}_T^h(t) = (\mathcal{D} * E\varphi^h)(t) \qquad t < T + \delta,$$

  which closes the loop.

Theorem 8.5.2 (Mapping properties of the Galerkin solver for $\mathcal{W}$)**.** *If $\alpha \in \mathcal{W}_+^1(\mathbb{R}; H^{-1/2}(\Gamma))$, then*

$$\mathcal{D} * \mathcal{G}_\varphi^h * \alpha \in \mathcal{C}_+^1(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}_+(\mathbb{R}; H^1(\mathbb{R}^d \setminus \Gamma))$$

and therefore

$$\mathcal{G}_\varphi^h * \alpha \in \mathcal{C}_+(\mathbb{R}; H^{1/2}(\Gamma)).$$

Moreover, for all $t \ge 0$,

$$\|(\mathcal{D} * \mathcal{G}_\varphi^h * \alpha)(t)\|_{1,\mathbb{R}^d\setminus\Gamma} \le C\left(|\rho(t)| + H_2(\partial^{-1}\alpha, t\,|\,H^{-1/2}(\Gamma))\right),$$

where

$$\rho(t) := \int_0^t (t-\tau)\langle \alpha(t), 1\rangle_\Gamma \mathrm{d}\tau.$$

8.6 Galerkin error operator

In this section the input is a causal $H^{1/2}(\Gamma)$-valued distribution φ. We then consider the causal solution of

$$\varphi^h \in Y_h \qquad \mathcal{W} * (\varphi^h - \varphi) \in Y_h^\circ,$$

and the associated potential

$$\varepsilon^h := \mathcal{D} * (\varphi^h - \varphi) = \mathcal{D} * \mathcal{E}_\varphi^h * \varphi,$$

which is characterized by the problem

$$\begin{aligned}
\varepsilon^h \in \mathrm{TD}(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)),\\
\ddot{\varepsilon}^h = \Delta\varepsilon^h,\\
[\![\gamma\varepsilon^h]\!] + \varphi \in Y_h^\circ,\\
[\![\partial_\nu\varepsilon^h]\!] = 0,\\
\partial_\nu\varepsilon^h \in Y_h.
\end{aligned}$$

The results of this section are given without proof. The reader is suggested to check all the results. After cut-off, the **dynamical system triplet** is the one of Section 8.5. The **lifting** $L : H^{1/2}(\Gamma) \to \mathbb{Y}_T$ is given by the solution of the variational problem:

$$\left[\begin{array}{l} u_0 \in H^1_{\partial B_T}(B_T \setminus \Gamma), \quad [\![\gamma u_0]\!] + \varphi \in Y_h^\circ, \\ (\nabla u_0, \nabla v)_{B_T\setminus\Gamma} + (u_0, v)_{B_T} = 0 \qquad \forall v \in V, \end{array}\right. \tag{8.44}$$

yielding the bound (see (8.28))

$$\|L\varphi\|_{1,B_T\setminus\Gamma} = |L\varphi|_{\mathbb{Y}_T} \leq C_\gamma^\dagger \|\varphi\|_{1/2,\Gamma}$$

It is important to note that

$$PL\varphi$$

(take $v = \chi_{\Omega_-}$ in (8.44)), which will eliminate the need of taking care of rigid motions in the **associated evolution problem in a bounded domain**. Taking now $\varphi \in \mathcal{C}_0^3([0,\infty); H^{1/2}(\Gamma))$, defining

$$u_0(t) := L\varphi(t),$$

and $v_0 : [0,\infty) \to D(A)$ as the solution of

$$\ddot{v}_0(t) = \Delta v_0(t) + L(\varphi(t) - \ddot{\varphi}(t)) \qquad t \geq 0, \qquad v_0(0) = \dot{v}_0(0) = 0,$$

we can build the function

$$\varepsilon_T^h := u_0 + v_0 \in \mathcal{C}^2([0,\infty); L^2(B_T)) \cap \mathcal{C}^1([0,\infty); H^1(B_T \setminus \Gamma)) \cap \mathcal{C}([0\infty); \mathbb{Y}_T))$$

that solves

$$\begin{aligned} \ddot{\varepsilon}_T^h(t) &= \Delta \varepsilon_T^h(t) & \quad \forall t \geq 0, \\ \gamma_T \varepsilon_T^h(t) &= 0 & \quad \forall t \geq 0, \\ [\![\gamma \varepsilon_T^h]\!](t) + \varphi(t) &\in Y_h & \quad \forall t \geq 0, \\ [\![\partial_\nu \varepsilon_T^h]\!](t) &= 0 & \quad \forall t \geq 0, \\ \partial_\nu \varepsilon_T^h(t) &\in Y_h^\circ & \quad \forall t \geq 0, \\ \varepsilon_T^h(0) = \dot{\varepsilon}_T^h(0) &= 0. \end{aligned}$$

Here are the resulting bounds for $t \geq 0$:

$$\begin{aligned} \|\varepsilon_T^h(t)\|_{1,B_T\setminus\Gamma} &\leq C_\gamma^\dagger (\|\varphi(t)\|_{1/2,\Gamma} + \widehat{c}(T)\, B(\varphi, t)), \\ \|\dot{\varepsilon}_T^h(t)\|_{1,B_T\setminus\Gamma} &\leq C_\gamma^\dagger (\|\dot{\varphi}(t)\|_{1/2,\Gamma} + B(\varphi, t) + B(\dot{\varphi}, t)), \\ \|\Delta\varepsilon_T^h(t)\|_{B_T\setminus\Gamma} &\leq C_\gamma^\dagger (\|\varphi(t)\|_{1/2,\Gamma} + 2B(\dot{\varphi}, t)). \end{aligned}$$

What is left follows the usual pattern of identification of the Galerkin error operator as the extension $\underline{\varepsilon}_T^h$ for $t < T + \delta$ and then use of a density argument to relax the continuity hypotheses on the side of φ. This is the result that comes out of this process.

Theorem 8.6.1 (Mapping properties of the Galerkin error operator for $\mathcal{W}$)**.** *If* $\varphi \in \mathcal{W}^2_+(\mathbb{R}; H^{1/2}(\Gamma))$*, then*

$$\mathcal{D} * \mathcal{E}^h_\varphi * \varphi \in \mathcal{C}^1_+(\mathbb{R}; H^1(\mathbb{R}^d \setminus \Gamma)) \cap \mathcal{C}_+(\mathbb{R}; L^2(\mathbb{R}^d))$$

and therefore

$$\mathcal{E}^h_\varphi * \varphi \in \mathcal{C}_+(\mathbb{R}; H^{1/2}(\Gamma)).$$

Moreover, for all $t \geq 0$

$$\|(\mathcal{D} * \mathcal{E}^h_\varphi * \varphi)(t)\|_{1,\mathbb{R}^d\setminus\Gamma} \leq C\, H_3(\partial^{-1}\varphi, t \,|\, H^{1/2}(\Gamma)).$$

Final remark. It is not difficult to show then that the bound of Theorem 8.6.1 is actually an error estimate for Galerkin semidiscretization. If $\Pi^h : H^{1/2}(\Gamma) \to Y_h$ is the best approximation operator on Y_h, then

$$\|u^h(t) - u(t)\|_{1,\mathbb{R}^d\setminus\Gamma} \leq C\, H_3(\partial^{-1}(\varphi - \Pi^h\varphi), t \,|\, H^{1/2}(\Gamma)),$$

which follows from the fact that $\mathcal{E}^h_\varphi * (\varphi - \Pi^h\varphi) = 0$ for all φ.

8.7 Exercises

1. (Section 8.2) For the space $H_h := \{u \in H^1(\mathbb{R}^d \setminus \Gamma) \,:\, [\![\gamma u]\!] \in Y_h\}$, prove that the operator $H_h \to H^{1/2}(\Gamma) \to Y_h$ given by
$$u \mapsto (\gamma^- u, [\![\gamma u]\!])$$
is surjective.

2. (Section 8.2) Prove Lemma 8.2.1 (**Hint.** Use the identity
$$(\nabla u, \nabla v)_{\mathbb{R}^d\setminus\Gamma} + (\Delta u, v)_{\mathbb{R}^d\setminus\Gamma} = \langle [\![\partial_\nu u]\!], \gamma^- v\rangle_\Gamma + \langle \partial^+_\nu u, [\![\gamma v]\!]\rangle_\Gamma$$
valid for all $u \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma)$ and $v \in H^1(\mathbb{R}^d \setminus \Gamma)$, and apply the result ofthe previous exercise.)

3. (Section 8.2) Prove (8.16).

4. (Section 8.2) Prove (8.20).

5. (Section 8.2) Compare the bound (8.21) with the result of estimating
$$\|\mathrm{E}^h_\varphi(s)\|_{H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)} \leq 1 + \|\mathrm{G}^h_\varphi(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}\|\mathrm{W}(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}.$$

6. (Section 8.2) The Laplace domain bounds of Section 8.2 provide time domain bounds for the Galerkin solver and error operator associated to the semidiscretization of the equation $\mathcal{W} * \varphi = \alpha$. Write them down.

7. (Section 8.4) Prove the bounds (8.30).

8. (Section 8.5) Prove the bounds of Proposition 8.5.1.

9. (Section 8.5) Prove Theorem 8.5.2.

10. (Section 8.6) Prove the results of Section 8.6.

Chapter 9

Full discretization revisited

9.1 Convolution Quadrature in the Laplace domain

9.2 Examples of time domain techniques

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