MATH 353: Engineering Mathematics III – Section 012

Spring 2014 (F.–J. Sayas) Homework #10 Due May 16

Important. Whenever you write a function, don't forget to include the help lines. The axes in all plots have to be labeled. Absolutely no late homework.

1. (By hand + Computer – 5 points) We have the following matrices

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 3 \\ 4 & 5 & 7 \end{bmatrix}, \qquad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}, \qquad \mathrm{U} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

   that satisfy $\mathrm{A} = \mathrm{LU}$. (You do not need to check this.) Let now

$$\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

   Solve the following systems by hand

$$\mathrm{L}\mathbf{y} = \mathbf{b} \qquad \text{and then} \qquad \mathrm{U}\mathbf{x} = \mathbf{y}.$$

   Compare it with the Matlab solution of the system $\mathrm{A}\mathbf{x} = \mathbf{b}$.

2. (By hand – 5 points) Give a mathematical argument that shows that if $\mathrm{A} = \mathrm{LU}$ and you solve the triangular systems $\mathrm{L}\mathbf{y} = \mathbf{b}$ and $\mathrm{U}\mathbf{x} = \mathbf{y}$, then you have solved the system $\mathrm{A}\mathbf{x} = \mathbf{b}$.

3. (By hand – 5 points) Assume that we have a matrix A and we have been able to write

$$\mathrm{PA} = \mathrm{LU},$$

   where

   - P is a permutation matrix
   - L is a lower triangular matrix
   - U is an upper triangular system.

   What can you do to solve a system $\mathrm{A}\mathbf{x} = \mathbf{b}$ if you know the above decomposition? (**Hint.** Think of the equivalent system $\mathrm{PA}\mathbf{x} = \mathrm{P}\mathbf{b}$ and use the idea of the previous exercise.)

4. (By hand + Computer – 5 points) Consider the matrix

$$\mathrm{Q} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1\sqrt{6} & -2/\sqrt{6} \end{bmatrix}.$$

Show (by hand) that it is an orthogonal matrix. Solve by hand the system $Q\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(**Hint.** If you know what you have to do, this takes two minutes.) Compare it with the Matlab solution of the system.

5. (By hand + Computer – 5 points) Here's a funny example I found in the wikipedia

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix} = QR$$

where

$$Q = \begin{bmatrix} 6/7 & -69/175 & -58/175 \\ 3/7 & 158/175 & 6/175 \\ -2/7 & 6/35 & -33/35 \end{bmatrix} \qquad R = \begin{bmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & 35 \end{bmatrix}.$$

(a) Use Matlab to verify that $A = QR$ and that Q is an orthogonal matrix.

(b) Consider the system

$$R\mathbf{x} = Q^\top \mathbf{b} \qquad \text{where} \qquad \mathbf{b} = \begin{bmatrix} -102 \\ 544 \\ 167 \end{bmatrix}.$$

Solve it by hand. Compare it with the Matlab solution.

(c) Give an argument that shows that the solution of a system $R\mathbf{x} = Q^\top \mathbf{b}$ is the same as the solution of $A\mathbf{x} = \mathbf{b}$ when $A = QR$ and Q is an orthogonal matrix.

6. (Computer – 15 points) Modify the function `heatForwardFD.m` given in Lab #11 so that it becomes

```
[U,x,t]=heatreactionForwardFD(D,R,interval,T,u0,left,right,N,M)
```

in order to solve the reaction-diffusion problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - R\,u, \qquad a < x < b, \quad 0 < t \le T,$$

($D > 0$ is the diffusivity, $R \ge 0$ is the reaction parameter), with initial condition at time $t = 0$

$$u(x,0) = u_0(x) \qquad a \le x \le b,$$

and two boundary conditions at $x = a$ and $x = b$ for all times

$$u(a,t) = l(t), \qquad u(b,t) = r(t) \qquad 0 < t \le T.$$

Using the notations of the Lab, the time-stepping process computes for $n \ge 0$:

$$\frac{U_i^{n+1} - U_i^n}{k} = \mathrm{D}\frac{U_{i-1}^n - 2U_i^n + U_{i-1}^n}{h^2} - R\,U_i^n \qquad i = 1,\ldots,N,$$

so you can easily figure out what the process is. You need to provide:

(a) The new code. The modifications are minimal!

(b) A script (modify `scriptMay9.m`) to solve in the interval $(0, 2)$, with $T = 1$, $u_0(x) = x^2(2 - x)$, $D = 0.3$, $R = 0.1$, $l(t) = \sin(t)$, $r(t) = \frac{1}{2}\sin(t)$. You will need to choose the discretization parameters N and M so that the method is stable.

(c) A snapshot of the solution at time $t = T$ computed with N, M. On top of that show the solution computed with $N, 2M$. Be sure that you get very similar solutions. Otherwise, you might have a bug in your code.