MATH 353: Engineering Mathematics III – Section 012

Spring 2014 (F.–J. Sayas) | Homework #7 | Due April 18

Important. Whenever you write a function, don't forget to include the help lines. The axes in all plots have to be labeled and whenever there are several graphs, there should be a legend explaining which is which.

1. (Review - 5 points) Find coefficients α and β so that

$$\frac{\alpha f(x_0+h) + (-\alpha-\beta) f(x_0) + \beta f(x_0 - \frac{h}{2})}{h} = f'(x_0) + \mathcal{O}(h^2).$$

2. (Review – 5 points) We consider the following numerical integration formula:

$$\int_a^b f(t)\mathrm{d}t = (b-a) f(a).$$

What is the degree of precision of this formula?

3. (Computer - review – 5 points) For this exercise you need to use your code for the trapezoidal rule and the Simpson rule. For the integral

$$\int_0^1 f(x)\mathrm{d}x, \qquad f(x) = x^2\, e^x,$$

I want you to check that

$$\tfrac{4}{3} T_{h/2}(f) - \tfrac{1}{3} T_h(f) = S_h(f).$$

Here $T_h(f)$ and $S_h(f)$ are the trapezoidal and Simpson rules, respectively, applied with $h = (b-a)/m$, that is, with m partitions. Therefore $T_{h/2}(f)$ is the trapezoidal rule applied with $2m$ partitions. Check that this formula holds for at least three values of m.

4. (By hand – use calculator – 5 points) Take two steps of length $h = 0.01$ with the midpoint method for the initial value problem

$$y' = \frac{t}{1+y^2}, \quad 1 \le t, \qquad y(1) = 0.$$

Use all the possible digits from your calculator, **but**, at the end, give the result showing only four significant digits.

5. (By hand – use calculator – 5 points) We have used two different methods to compute the solution of an initial value problem:

$$y' = f(t,y), \quad a \le t \le b, \qquad y(a) = y_a.$$

We have done this for $n = 10, 20, 40, 80$ and 160 time steps. Then we have computed the errors:

$$E_h^{\text{Meth\#1}} = \max_{0\le j\le n} |w_j^{\text{Meth\#1}} - y(t_j)| \qquad \text{and} \qquad E_h^{\text{Meth\#2}} = \max_{0\le j\le n} |w_j^{\text{Meth\#2}} - y(t_j)|.$$

These are the errors.

Errors Method 1	Errors Method 2
0.015207614655732	0.121784073781251
0.004037913819598	0.059953565850001
0.001037371962928	0.028020725164368
0.000262757612076	0.013566946929654
0.000066134187187	0.006677021808841

Here is what you have to explain: one of the methods is Euler's method and the other one is the midpoint method. Which is which and why?

6. (Computer – 5 points) The solution of the initial value problem

$$y' = t\,y, \quad 0 \le t, \qquad\qquad y(0) = 3,$$

is $y(t) = 3\,e^{\frac{t^2}{2}}$. Using Heun's method (the code is in the website) in the interval $[0, 4]$ and at least 10 different values of n (the number of time steps), check that the order of Heun's method (the explicit trapezoidal method) is two, that is,

$$\max_{0 \le j \le n} |w_j - y(t_j)| = \mathcal{O}(h^2).$$

The output for this problem should be: your code and a loglog plot with the errors. Make sure that it is clear what are the errors in the graph and how you compare with a line to see that the order is actually two.

7. (Computer – 10 points) We want to study another method to solve numerically

$$y' = f(t, y), \qquad a \le y \le b, \qquad\qquad y(a) = y_a.$$

We start as usual with $w_0 = y_a$. For the next time-steps we use three internal stages:

$$\begin{aligned} k_1 &= f(t_i, w_i), \\ k_2 &= f(t_i + \tfrac{1}{2}h, w_i + \tfrac{1}{2}h\,k_1), \\ k_3 &= f(t_i + h, w_i - hk_1 + 2hk_2), \\ w_{i+1} &= w_i + \tfrac{h}{6}(k_1 + 4k_2 + k_3). \end{aligned}$$

This method is called Runge's method. We are going to test it on the equation

$$y' + t\,y^2 = 0, \qquad 0 \le t \le 2, \qquad y(0) = 1,$$

whose solution is $y(t) = \frac{2}{2+t^2}$.

(a) Write a program for this method following the format of Heun's method we have programmed in class.

(b) Run the code for the above problem, using 10 time steps. Make a graph with the numerical solution and plot the exact solution on top of it.

(c) Using several values of h, a loglog plot and whatever you find appropriate, find p such that

$$\max_{0 \le j \le n} |w_j - y(t_j)| = \mathcal{O}(h^p).$$