MATH 353
Engineering mathematics III

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LAGRANGE INTERPOLATION

A simple problem

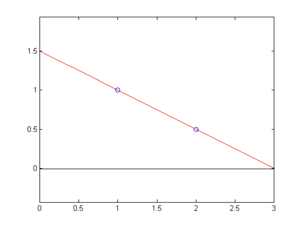
Find the equation for the line going through

$$(x_1, y_1) \qquad \text{and} \qquad (x_2, y_2)$$

Here's the solution:

$$y = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}.$$

- How do we know? Just check that the result is correct!
- Note that there's only one line going through two points



Another look

Here's the same equation again

$$y = y_1 \underbrace{\frac{x - x_2}{x_1 - x_2}}_{L_1(x)} + y_2 \underbrace{\frac{x - x_1}{x_2 - x_1}}_{L_2(x)}$$

In red we emphasize data. Look now at the polynomials L_1 and L_2:

$$\begin{array}{ll} L_1(x_1) = 1 & \qquad L_1(x_2) = 0, \\ L_2(x_1) = 0 & \qquad L_2(x_2) = 1. \end{array}$$

The polynomials L_1 and L_2 depend on where we have data (x_1 and x_2), but they do not use the y values in their definition.

A slightly more complicated problem

Find the equation for the parabola going through

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \text{and} \qquad (x_3, y_3)$$

First idea (not a good one): write

$$y = a + bx + cx^2,$$

and use the conditions

$$y_i = a + bx_i + cx_i^2 \qquad i = 1, 2, 3$$

to find the coefficients a, b, c. There's a better idea in the next slide. (It's such a good idea, that it's the solution to the problem.)

The solution

$$P(x) = y_1 \underbrace{\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}}_{L_1(x)} + y_2 \underbrace{\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}}_{L_2(x)} + y_3 \underbrace{\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}}_{L_3(x)}$$

Check that this is the solution! Actually, check the following

$$\begin{array}{lll} L_1(x_1) = 1 & L_1(x_2) = 0 & L_1(x_3) = 0, \\ L_2(x_1) = 0 & L_2(x_2) = 1 & L_2(x_3) = 0, \\ L_3(x_1) = 0 & L_3(x_2) = 0 & L_3(x_3) = 1. \end{array}$$

Apply to a particular example

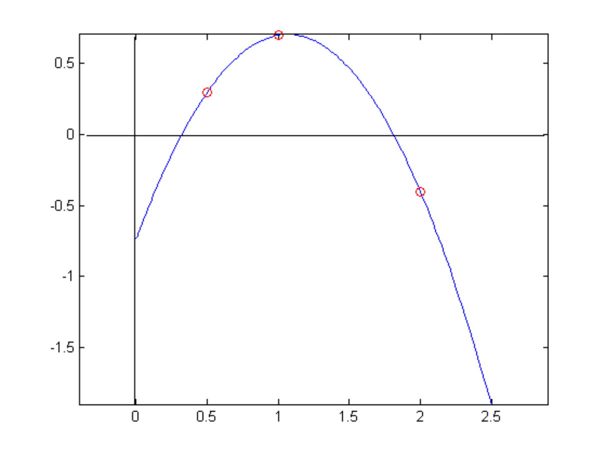
The points are $(0.5, 0.3)$, $(1, 0.7)$ and $(2, -0.4)$.

$$P(x) = 0.3 \underbrace{\frac{(x-1)(x-2)}{(0.5-1)(0.5-2)}}_{L_1(x)} + 0.7 \underbrace{\frac{(x-0.5)(x-2)}{(1-0.5)(1-2)}}_{L_2(x)} - 0.4 \underbrace{\frac{(x-0.5)(x-1)}{(2-0.5)(2-1)}}_{L_3(x)}$$

You can simplify this formula if you are willing to waste your time. We will not do it (in principle, you shouldn't do it unless especifically requested!). The formula for $P(x)$ is complicated but the computer will not care.

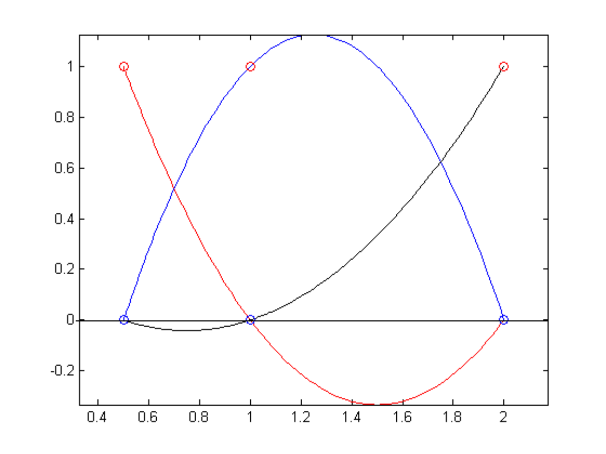
A figure of what we just did: interpolation

The points are $(0.5, 0.3)$, $(1, 0.7)$ and $(2, -0.4)$.



Here are the three polynomials L_1, L_2, L_3

The relevant values are $x_1 = 0.5$, $x_2 = 1$ and $x_3 = 2$. Can you see which one is L_1?



Let's think about this problem

Given three points

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \text{and} \qquad (x_3, y_3)$$

with x_1, x_2 and x_3 distinct, we found a polynomial $P(x)$ of degree 2 (or less) such that

$$P(x_1) = y_1, \qquad P(x_2) = y_1, \qquad \text{and} \qquad P(x_3) = y_3.$$

Are there more polynomials (of the same degree or less) satisfying these conditions? The answer is no. Here's why! If Q also satisfied this conditions, then

$$R(x) = P(x) - Q(x)$$

is a polynomial of degree two or less satisfying

$$R(x_1) = 0, \qquad R(x_2) = 0, \qquad R(x_3) = 0.$$

A quadratic polynomial cannot have three roots unless it is the zero polynomial. In other words, $R(x) = 0$ and $P(x) = Q(x)$. Our problem has a unique solution and we just gave the formula for it.

General polynomial interpolation problem

Lagrange polynomial interpolation problem

Given points

$$(x_i, y_i), \qquad i = 1, \ldots, n+1$$

with distinct x_i, find a polynomial $P(x)$ of degree n or less satisfying

$$P(x_i) = y_i \qquad i = 1, \ldots, n+1.$$

Theorem. The Lagrange polynomial interpolation problem above has a unique solution.

Important argument. We will give a formula for the solution. The argument showing that the solution is unique is similar to what we did in the quadratic case.

The Lagrange formula...

... for the Lagrange interpolation problem

Define the polynomials (for $k = 1, \ldots, n+1$)

$$L_k(x) = \frac{x - x_1}{x_k - x_1} \cdots \frac{x - x_{k-1}}{x_k - x_{k-1}} \frac{x - x_{k+1}}{x_k - x_{k+1}} \cdots \frac{x - x_{n+1}}{x_k - x_{n+1}}.$$

The polynomials have degree exactly n and satisfy

$$L_k(x_i) = \begin{cases} 1 & i = k, \\ 0, & i \neq k \end{cases}$$

Then

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \ldots + y_{n+1} L_{n+1}(x)$$

is the solution to the interpolation problem.

Towards the code

We want to evaluate

$$y_k \frac{x - x_1}{x_k - x_1} \cdots \frac{x - x_{k-1}}{x_k - x_{k-1}} \frac{x - x_{k+1}}{x_k - x_{k+1}} \cdots \frac{x - x_{n+1}}{x_k - x_{n+1}}.$$

We want to be able to do this when x contains a lot of valus of the x variable in form of an array. Here's how we loop...

```
c=y(k);
for j=[1:k-1 , k+1:n+1]
    c=c.*(X-x(j))./(x(k)-x(j));
end
```

Note that

```
[1:k-1 , k+1:n+1]
```

is the list of numbers from 1 to $n+1$ missing k

```
function Y=evaluatelagrange(x,y,X)

% Y=evaluatelagrange(x,y,X)
%
% Input:
%       x    : vector with n+1 entries (points x_i)
%       y    : vector with n+1 entries (points y_i)
%       X    : points where we want to evaluate
% Output:
%       Y    : values of the interpolant P at the points X

Y=0;
n=length(x)-1;
for k=1:n+1
    c=y(k);
    for j=[1:k-1 , k+1:n+1]      % list of 1 to n+1 w/o k
        c=c.*(X-x(j))./(x(k)-x(j));
    end
    Y=Y+c;
end
return
```