MATH 353: Engineering Mathematics III – Section 012

Spring 2013 (F.–J. Sayas) Homework #7 Due April 12

Important. Whenever you write a function, don't forget to include the help lines. The axes in all plots have to be labeled.

1. (Review) We consider the following numerical integration formula:

$$\int_a^b f(t)\mathrm{d}t = (b-a)f(a).$$

What is the degree of precision of this formula?

2. (Computer - review) For this exercise you need to use your code for the trapezoidal rule and the Simpson rule. For the integral

$$\int_0^1 f(x)\mathrm{d}x, \qquad f(x) = x^2\,e^x,$$

I want you to check that

$$\tfrac{4}{3}T_{h/2}(f) - \tfrac{1}{3}T_h(f) = S_h(f).$$

Here $T_h(F)$ and $S_h(f)$ are the trapezoidal and Simpson rules, respectively, applied with $h = (b-a)/m$, that is, with m partitions. Therefore $T_{h/2}(f)$ is the trapezoidal rule applied with $2m$ partitions. Check that this formula holds for at least three values of m.

3. (By hand – use calculator) Take two steps of length $h = 0.01$ with the midpoint method for the initial value problem

$$y' = \frac{t}{1+y^2}, \quad 1 \le t, \qquad y(1) = 0.$$

Use all the possible digits from your calculator, **but**, at the end, give the result showing only four significant digits.

4. (By hand – use calculator) We have used two different methods to compute the solution of an initial value problem:

$$y' = f(t,y), \quad a \le t \le b, \qquad y(a) = y_a.$$

We have done this for $n = 10, 20, 40, 80$ and 160 time steps. Then we have computed the errors:

$$E_h^{\mathrm{Meth}\#1} = \max_{0\le j\le n} |w_j^{\mathrm{Meth}\#1} - y(t_j)| \qquad \text{and} \qquad E_h^{\mathrm{Meth}\#2} = \max_{0\le j\le n} |w_j^{\mathrm{Meth}\#2} - y(t_j)|.$$

These are the errors.

```
Errors Method 1      Errors Method 2
0.015207614655732    0.121784073781251
0.004037913819598    0.059953565850001
0.001037371962928    0.028020725164368
0.000262757612076    0.013566946929654
0.000066134187187    0.006677021808841
```

Here is what you have to explain: one of the methods is Euler method and the other one is the midpoint method. Which is which and why?

5. (Computer) The solution of the initial value problem

$$y' = t\,y, \quad 0 \le t, \qquad\qquad y(0) = 3,$$

is $y(t) = 3\,e^{\frac{t^2}{2}}$. Using Heun's method (the code is in the website) in the interval $[0,4]$ and at least 10 different values of n (the number of time steps), check that the order of Heun's method (the explicit trapezoidal method) is two, that is,

$$\max_{0\le j\le n} |w_j - y(t_j)| = \mathcal{O}(h^2).$$

The output for this problem should be: your code and a loglog plot with the errors. Make sure that it is clear what are the errors in the graph and how you compare with a line to see that the order is actually two.

6. (Computer) We want to study another method to solve numerically

$$y' = f(t,y), \qquad a \le y \le b, \qquad\qquad y(a) = y_a.$$

We start as usual with $w_0 = y_a$. For the next time-steps we use three internal stages:

$$\begin{aligned} k_1 &= f(t_i, w_i), \\ k_2 &= f(t_i + \tfrac{1}{2}h, w_i + \tfrac{1}{2}h\,k_1), \\ k_3 &= f(t_i + h, w_i - hk_1 + 2hk_2), \\ w_{i+1} &= w_i + \tfrac{h}{6}(k_1 + 4k_2 + k_3). \end{aligned}$$

This method is called Runge's method. We are going to test it on the equation

$$y' + t\,y^2 = 0, \qquad 0 \le t \le 2, \qquad y(0) = 1,$$

whose solution is $y(t) = \frac{2}{2+t^2}$.

(a) Write a program for this method following the format of Heun's method we have programmed in class.

(b) Run the code for the above problem, using 10 time steps. Make a graph with the numerical solution and plot the exact solution on top of it.

(c) Using several values of h, a loglog plot and whatever you find appropriate, find p such that

$$\max_{0\le j\le n} |w_j - y(t_j)| = \mathcal{O}(h^p).$$

7. (By hand) **A tricky, but not difficult, question.** The solution of the following initial value problem

$$y' = f(t), \quad 0 \le t \le 1, \qquad y(0) = 0$$

is an antiderivative of f. In particular

$$y(1) = \int_0^1 f(\tau) \mathrm{d}\tau.$$

Show that if you apply the midpoint method to the above differential equation, with n time steps, at the last time step you have

$$w_n = h \sum_{j=0}^{n-1} f(t_j + \tfrac{h}{2}) \approx y(t_n) = y(1).$$

How is this related to what we did in the previous chapter? (That is, can you recognize any method for numerical integration you have seen?) Repeat the argument with:

(a) The trapezoidal method.

(b) Runge's method in problem 6.