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MATH 508: Introduction to Complex Variables and Apps

Spring'12 Quiz # 5 May 10

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1. (10 points) Consider the function
$$f(z) = \frac{\cos z}{z^2+1}.$$
Assume that we have computed its Taylor series at $z_0 = 3$ (don't compute the series!). What is the radius of convergence of this series? (**Hint.** The Taylor series defines an analytic function in a disk and coincides with f in its domain of convergence.)

2. (10 points) If
$$f(z) = \sum_{k=0}^{\infty} (k+3) z^{2k},$$
what are $f(0)$, $f'(0)$ and $f''(0)$? (**Hint.** Be careful with the powers of z involved in the series.)

3. (10 points) The power series
$$f(z) = \sum_{k=0}^{\infty} (k+1)^2 z^k$$
has radius the convergence equal to one. Compute
$$\oint_{|z|=1/2} \frac{f(z)}{z^3}\, \mathrm{d}z.$$

4. (10 points) Find the Laurent series for $z^2\, e^{1/z}$ in $|z| > 0$. What type of singularity is $z = 0$?

5. (10 points) Consider the function
$$f(z) = \frac{2z + 2\imath - 1}{(z-1)(z+2\imath)} = \frac{1}{z-1} + \frac{1}{z+2\imath}.$$
Find its Laurent series for $1 < |z| < 2$. Use the following trick for your computations:
$$\frac{1}{z-a} = \begin{cases} -\dfrac{1}{a}\dfrac{1}{1-\frac{z}{a}} = -\dfrac{1}{a}\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{a^n}, & |z| < |a|, \\ \dfrac{1}{z}\dfrac{1}{1-\frac{a}{z}} = \dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} \frac{a^n}{z^n}, & |z| > |a|. \end{cases}$$

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Spring'12 Quiz # 4 April 24

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1. (10 points) Let Γ be part of the circumference $|z| = 2$ that lies in the half plane $\text{Re}z > 0$, with counterclockwise orientation. **Use the definition of integral with a parametrization of** Γ, to compute

$$\int_{\Gamma} z^2 \mathrm{d}z.$$

Compare the result with what you would obtain using an antiderivative of z^2 and the result on independence of path.

2. (10 points) Let Γ be the line segment starting in $z = \imath$ and ending in $z = \pi$. Compute

$$\int_{\Gamma} \sin^2 z \cos z \mathrm{d}z.$$

3. (10 points) Make a plot of the set $|\text{Re}z| < 2$. Is it simply connected? (Justify your answer.)

4. (10 points) Let C be the circle $|z| = 3$ traversed once **clockwise** and consider the function $f(z) = \text{Log}(z + 4)$. Determine the domain of analyticity of f and compute

$$\int_{C} \text{Log}(z + 4) \mathrm{d}z.$$

5. (10 points) Let C be the circle $|z| = 2$ traversed once in the positve sense. Compute the following integral:

$$\int_{C} \frac{4z^2 + \imath z}{(z + \imath)^3} \mathrm{d}z.$$

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Spring'12 Quiz # 3 April 10

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1. (10 points) In this exercise we want to parameterize the line segment γ from $z = 2-\imath$ to $z = 1+\imath$.
   (a) Construct a parameterization $z_1 : [0,1] \to \gamma$.
   (b) Construct another parameterization the same segment $z_2 : [2,5] \to \gamma$. What is $z_2'(t)$?

2. (10 points) Evaluate $\int_0^2 \frac{t}{(t^2+\imath)^2}\mathrm{d}t$. (**Hint.** What is the derivative of $(t^2+\imath)^{-1}$?)

3. (15 points) Let Γ be the smooth open arc parameterized by $z(t) = e^{\imath t}$ with $-\pi/2 \le t \le 0$ and let $f(z) = 2z+1$.
   (a) Plot the contour Γ (including information about orientation, starting and end-points).
   (b) Evaluate $\int_\Gamma f(z)\mathrm{d}z$.

4. (15 points) Use the formula

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \max_{z\in\gamma} |f(z)| \times \text{length}(\gamma)$$

to show that

$$\left|\int_\gamma \operatorname{Log} z \, \mathrm{d}z\right| \le \frac{\pi^2}{4}$$

where γ is the arc of the circle $|z| = 1$ that lies in the first quadrant. (**Hint.** For points on the circle $|z| = 1$, there is a very simple formula for $|\operatorname{Log} z|$.)

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Spring'12 Quiz # 2 $\frac{1}{2}$ March 15

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1. Make a good plot (in separate figures) of the following sets. Mark with continuous lines the parts of their boundary (border) that are contained in the set and with dashed lines those that are not. Mark relevant points in the graph.

   (a) Plot the set D of those z such that $1 < |z| < 2$ and $-\frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{\pi}{4}$.

   (b) Plot the set of the points z^2 where $z \in D$.

   (c) Plot the set $z^{1/2}$, where $z \in D$. (This last one is tricky, because we have two square roots for each number).

2. What is $\log e^{1+\imath}$? (Don't rush your answer. There's more than one logarithm for each number.)

3. Consider all the points $|z| = e$, the points $\text{Log} z$ and $\log z$. Plot where they are. (Again, each point has many logarithms but only one Logarithm.)

4. We have a rational function

$$R(z) = \frac{z}{(1 - z^2)(1 + z^2)^3}.$$

   What is the form of its partial fraction decomposition? Do NOT compute the coefficients, just show the form of the decomposition.

5. Make a plot (including arrows for directions) of the points $z = 1 + e^{-\imath t}$ where $-\pi < t < \pi$. A computation we did for one homework assignment shows that for those z, $\text{Re}\frac{1}{z} = \frac{1}{2}$. Can you repeat the computation? Finally, make a plot of $\frac{1}{z}$ for $z = 1 + e^{-\imath t}$, with an arrow showing the direction as t increases.

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Spring'12 Quiz # 2 March 6

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1. (5 points) Show that $|e^z| \leq 1$ if $\operatorname{Re} z \leq 0$.
2. (5 points) Find all the values of $(-16)^{1/4}$. Plot them.
3. (10 points) Describe the range of the function $f(z) = z^2$ for z in the first quadrant, $\operatorname{Re} z \geq 0$, $\operatorname{Im} z \geq 0$.
4. (5 points) Find $\lim_{z\to\infty}(8z^3 + 5z + 2)$. (**Hint.** Write the function in the following form $z^3(8 + 5z^{-2} + 2z^{-3})$ and apply basic properties of limits.)
5. (10 points) Find the derivative of $f(z) = 6\imath(z^3 - 1)^4(z^2 + \imath z)^{100}$.
6. (5 points) Determine the points at which the function $\dfrac{\imath z^3 + 2z}{z^2 + 1}$ is not analytic.
7. (10 points) Show that if f is analytic in a disk D and $\operatorname{Re} f(x)$ is constant in D, then $f(z)$ must be constant in D. (**Hint.** Use the Cauchy–Riemann equations[1])

[1]**Cauchy–Riemann equations.** If we write $f(x + \imath y) = u(x, y) + \imath v(x, y)$ and f is analytic, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Also

$$f'(x + \imath y) = \frac{\partial u}{\partial x} + \imath\frac{\partial v}{\partial x}.$$

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Spring'12 Quiz # 1 February 21

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1. (5 points) Write $\dfrac{1-\imath}{3}$ in polar form, that is, write $\dfrac{1-\imath}{3} = r\, e^{\imath\theta}$.
2. (10 points) Show that $\overline{e^z} = e^{\overline{z}}$ for all z. (**Hint.** Start by writing $z = a + b\imath$ and look at both sides of the equality you are trying to prove.)
3. (5 points) Show that $e^{z+2\pi\imath} = e^z$ for all z.
4. (10 points) Find all the values of $\imath^{1/4}$. (Since you are at it, plot them.)
5. (5 points) Compute $\left|\dfrac{1+2\imath}{-2-\imath}\right|$ (**Hint.** You do not need to divide two numbers in order to know what the modulus of their quotient is.)
6. (5 points) Let $z = 2 + \imath$. Plot the points z, $-z$, $\overline{z}$, $-\overline{z}$ and $1/z$. (Make a single plot and mark any kind of alignment or geometric figure that shows up.)
7. (10 points) The zunction $t \mapsto 2\, e^{\imath t}$, $0 \le t \le 2\pi$, describes a circle of radius 2, traversed in the counterclowise direction and starting at the point $z = 2$. Describe the curve $z(t) = 2e^{\imath t} + \imath$ for $0 \le t \le 2\pi$. Plot it.

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Spring'12 | Quiz # 0 (not graded) | February 7

Unless asked to do otherwise, you can consider that you are done with a computation involving complex numbers once you have arrived at an expression

$$a + b\imath$$

with a and b real numbers.

1. Do these sums and subtractions:

$$(1+3\imath)+(2-4\imath) =$$

$$(2+\imath)+(2-\imath) =$$

$$(4+\imath)+2\imath =$$

$$(5-2\imath)-(3+\imath) =$$

2. Now these multiplications

$$(1+3\imath)(2+\imath) =$$

$$(1-3\imath)(2-5\imath) =$$

$$(2+3\imath)(2+3\imath) =$$

$$(3+\imath)^2 =$$

$$(4+4\imath)^2 \overset{?}{=} 16\,(1+\imath)^2 =$$

3. Compute

$$(1+\tfrac{1}{2}\imath)^2 \quad =$$

$$(1+\tfrac{1}{2}\imath)^3 \quad =$$

4. Show that for every a and b

$$(a+b\imath)(a-b\imath) = a^2+b^2$$

5. Do these divisions

$$\frac{1}{2+\imath} \quad =$$

$$\frac{3-\imath}{2-3\imath} \quad =$$

$$\frac{2+3\imath}{2\imath} \quad =$$

$$\frac{\frac{1}{2}+3\imath}{1+\frac{1}{2}\imath} \quad =$$

6. In the complex plane, draw the location of $1, \imath, -1$ and $-\imath$.

7. In the complex plane, draw the location of the numbers $3+2\imath$, $1+2\imath$, and $1-2\imath$.

8. In the complex plane, draw the location of $1+\frac{1}{2}\imath$, $(1+\frac{1}{2}\imath)^2$ and $(1+\frac{1}{2}\imath)^3$.

9. Draw a circle and, starting on the $x-$axis (the positive real axis of the complex plane), mark the following angles:

$$\frac{\pi}{2} \qquad \frac{\pi}{4} \qquad \pi \qquad \frac{5}{4}\pi \qquad \frac{\pi}{6}$$

Looking at the picture, say what the cosine and the sine of these angles are (some of them require some thinking).