NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2016 Final exam December 12

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- **Part of the grade of this exam evaluates your ability to write fully justified and readable proofs.**
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.**

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) Signature Date

Write here a three digit number

Write the same number

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
TOTAL	100 pts	

1. (20 points) **The discrete metric.** In this exercise you can only use basic definitions, but no theorems. Let X be an infinite set and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

   (a) Show that X is a metric space.

(b) Show that $E \subset X$ is compact if and only if E is finite.

(c) Show that $x_n \to x$ if and only if there exists N such that $x_n = x$ for all $n \geq N$.

(d) Show that X is complete.

2. (20 points) **Four questions about sequences.** Using only the definitions, prove the following results:

   (a) Every convergent sequence (in a metric space) is Cauchy.

(b) Every Cauchy sequence (in a metric space) is bounded.

(c) If $\{x_n\}$ and $\{y_n\}$ are convergent sequences in $\mathbb{C}$, then $\{x_n y_n\}$ is convergent.

(d) Show that if $\{x_n\}$ is a bounded increasing sequence in $\mathbb{R}$, then it is convergent. (Hint. You can figure out the limit.)

3. (20 points) **Dini's theorem.** Let X be a compact metric space and let $g_n : X \to \mathbb{R}$ be continuous functions satisfying

$$g_n(x) \geq g_{n+1}(x) \quad \forall x \in X, \forall n, \qquad \text{and} \qquad \lim_{n\to\infty} g_n(x) = 0 \quad \forall x \in X.$$

(a) For arbitrary $\varepsilon > 0$, consider the sets

$$K_n^\varepsilon = \{x \in X \,:\, g_n(x) \geq \varepsilon\}.$$

Show that

$$\bigcap_{n=1}^{\infty} K_n^\varepsilon = \emptyset$$

and therefore there exists N such that $K_n^\varepsilon = \emptyset$ for all $n \geq N$.

(b) Prove that $g_n \to 0$ uniformly in X.

4. (20 points) Let the functions $f_n : [0,1] \to [0,1]$ (for $n \in \mathbb{N}$) satisfy

$$\begin{aligned} f_n(0) &= 1 && \forall n, \\ f_n'(x) &\leq 0 && \forall n, \quad \forall x \in [0,1], \\ \lim_{n\to\infty} f_n(x) &= 0 && \forall x \in (0,1]. \end{aligned}$$

(a) Show that $\{f_n\}$ is uniformly convergent in $[\delta, 1]$ for all $\delta > 0$.

(b) Show that $\{f_n\}$ is not uniformly convergent in $[0, 1]$.

(c) Prove that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0.$$

5. (20 points) **The graph of a function.** Let X and Y be metric spaces and consider the metric

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

in $X \times Y$. Let $f : X \to Y$ be a function and

$$G = \{(x, f(x)) : x \in X\} \subset X \times Y$$

be its graph.

(a) Show that the sequence $\{(x_n, y_n)\}$ in $X \times Y$ is convergent if and only if the sequences $\{x_n\}$ and $\{y_n\}$ converge in X and Y respectively.

(b) Show that if f is continuous, then the graph is closed.

(c) Show that if Y is compact and G is closed, then f is continuous. (Hint. Given $x_n \to x$, show that there exists a subsequence such that $f(x_{n_k}) \to f(x)$.)

(d) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1/x & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that the graph of f is closed but f is not continuous.

6. (20 extra points) Let $G : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous, and consider the set

$$\mathcal{F} = \left\{ f(x) = \int_0^1 G(x, y) g(y) dy \; : \; g \in \mathcal{C}([0, 1]), \quad \int_0^1 |g(x)| dx \leq 1 \right\}.$$

Show that $\mathcal{F}$ is relatively compact in $\mathcal{C}([0, 1])$.

(Extra space for Question 6)