NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2016 | Midterm exam # 1 | October 5

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.**

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) Signature Date

Write here a three digit number

Write the same number

1	10 pts	
2	10 pts	
3	10 pts	
4	15 pts	
5	15 pts	
6	25 pts	
7	15 pts	
TOTAL	100 pts	

1. (10 points) Let $x, y \in \mathbb{R}$. Show that

$$\sup\{p + q \,:\, p, q \in \mathbb{Q}, \quad p < x, \quad q < y\} = x + y.$$

2. (10 points) Show that the set of irrational numbers

$$\{x \in \mathbb{R} \,:\, x \notin \mathbb{Q}\}$$

is uncountable.

3. (10 points) Let K_n for $n \in \mathbb{N}$ define a collection of compact non-empty sets with the property that

$$K_{n+1} \subset K_n \qquad \forall n \in \mathbb{N}.$$

Assume for the moment that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

Show that this assumption implies that

$$K_1 \subset \bigcup_{n=1}^{\infty} K_n^c$$

and continue the argument to prove that there exists m such that $K_1 \cap K_m = \emptyset$, and that this fact contradicts our assumption. State the theorem whose proof you have just given.

4. (15 points) Let $E \subset \mathbb{R}$, where $\mathbb{R}$ is equipped with the Euclidean metric. Show that if E is upper bounded and closed, then $\sup E \in E$.

5. (15 points) Let $E \subset X$, where X is a metric space and let E' be the set of its limit points. Show that E' is closed. (Hint. Consider the set $(E')'$.)

6. (25 points) Let X be a metric space, $x \in X$ and $r > 0$. Show that

$$\overline{N_r(x)} \subset \{y \in X \,:\, d(x,y) \leq r\}.$$

Show that the two sets are equal if $X = \mathbb{R}^k$ with the Euclidean distance and give an example (in the metric space of your choice) where the sets are not equal.

(Additional space for Problem 6)

7. (15 points) Let X be a metric space. The boundary of $E \subset X$ is defined as $\partial E = \overline{E} \cap \overline{E^c}$. Show that $\partial E = \overline{E} \setminus E^\circ = \{x \in \overline{E} : x \notin E^\circ\}$ and that E is closed if and only if $\partial E \subset E$.