NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2016 | Midterm exam # 2 | November 9

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.**

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) Signature Date

Write here a three digit number

Write the same number

1	10 pts	
2	10 pts	
3	10 pts	
4	10 pts	
5	15 pts	
6	15 pts	
7	15 pts	
8	15 pts	
TOTAL	100 pts	

1. (10 points) Let $E \subset X$, where X is a metric space and let E' be the set of its limit points. Show that E' is closed.

2. (10 points) Let $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$ (here $\mathbb{R}^k$ is endowed with the Euclidean metric). Show that

$$\partial N_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^k \,:\, |\mathbf{y} - \mathbf{x}| = r\}.$$

3. (10 points). We say that a subset E of a metric space is complete when every Cauchy sequence in E converges to an element of E. Show that if E is complete, then it is closed and that any closed subset of E is also complete.

4. (10 points) Let $\mathbf{f} : X \to \mathbb{R}^k$ be continuous on the metric space X and let $\mathbf{d}_1, \ldots, \mathbf{d}_m \in \mathbb{R}^k$ be given vectors. Show that

$$\{x \in X \,:\, \mathbf{f}(x) \cdot \mathbf{d}_\ell \leq 0 \quad \ell = 1, \ldots, m\}$$

is closed.

5. (15 points) Let $\{x_n\}$ be a sequence in a metric space X with the following property: there exists $x \in X$ such that every subsequence of $\{x_n\}$ contains a subsequence that converges to x. Show that $x_n \to x$.

6. (15 points) Let $f : X \to Y$ be a function defined between two metric spaces, and consider its graph

$$G = \{(x, f(x)) \, : \, x \in X\} \subset X \times Y$$

and the metric

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

defined in $X \times Y$.

(a) Show that if f is continuous, then the graph is closed. (Hint. Use sequences.)

(b) Show that if Y is compact and G is closed, then f is continuous. (Hint. Given $x_n \to x$, show that there exists a subsequence such that $f(x_{n_k}) \to f(x)$.)

7. (15 points) Let $f : (0, 1) \to \mathbb{R}$ be uniformly continuous.

   (a) Show that there exists $M > 0$ such that $|f(x)| < M$ for all x.

(b) Show that if $x_n \to 0$, then the sequence $f(x_n)$ is convergent as well.

(c) Show that $f(0_+)$ exists.

8. (15 points) Let $f : [0,1] \to \mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$. Assume that $f(0) = 0$ and there exists $C > 0$ such that

$$|f'(x)| \leq C|f(x)| \qquad \forall x \in [0,1].$$

(a) Show that for all $x \in (0,1)$

$$|f(x)| \leq \max_{c \in [0,x]} |f(c)| \, (C|x|)^n \qquad \forall n.$$

(b) Prove that f is identically zero.