Problems for Chapter 1

by F.J.Sayas, for MATH 600

September 4, 2018

Do not share these lists of problems outside the scope of the course. Problems listed as (R1.X) correspond to Rudin, Chapter 1, Problem X.

1. Let $p \neq q$, $p, q \in \mathbb{Q}$. Show that there are infinitely many rational numbers between them. (Hint. Add the midpoints.)

2. Consider the sets:
$$A = \{p \in \mathbb{Q} \,:\, p > 0, \quad p^2 < 3\}, \qquad B = \{p \in \mathbb{Q} \,:\, p > 0, \quad p^2 > 3\}.$$
Show that $\{p \in \mathbb{Q} \,:\, p > 0\} = A \cup B$ with disjoint union. Show that A and B are not-empty. Consider now the following algorithm: $p_1 = 1$ and $q_1 = 2$;
$$\begin{aligned} \text{for } n \geq 1 \quad & r_n = \tfrac{1}{2}(p_n + q_n), \\ & \text{if } r_n \in A, \text{ then} \quad p_{n+1} = r_n, \quad q_{n+1} = q_n, \\ & \text{if } r_n \notin A, \text{ then} \quad p_{n+1} = p_n, \quad q_{n+1} = r_n. \end{aligned}$$
Show that $p_n \in A$ for all n, $q_n \in B$ for all n and
$$|p_n - q_n| = \frac{1}{2^{n-1}} \quad \forall n.$$

3. Let F be an ordered field. Show that
$$0 < x < y \qquad \Longrightarrow \qquad x^n < y^n \quad \forall n \in \mathbb{N}.$$
(Hint. Show first that if $z > 0$, then $xz < yz$.)

4. (Based on (R1.5).) Let A be a non-empty lower bounded set of an ordered field with the least upper bound property. Show that the set
$$-A = \{-x \,:\, x \in A\}$$
is upper bounded and
$$\inf A = -\sup(-A).$$

5. (R1.2) Prove that there is no rational number p such that $p^2 = 12$.

6. (R1.8) Prove that no order can be defined in the complex field that makes it an ordered field.

7. (R1.6) [This problem is more challenging than the other ones.]