Problems for Chapter 2 (Part I)

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Do not share these lists of problems outside the scope of the course. Problems listed as (R2.X) correspond to Rudin, Chapter 2, Problem X.

1. **Some review.**

   (a) Show that given $x \in \mathbb{R}$

   $$x = \sup\{r \in \mathbb{Q} \,:\, r \le x\}.$$

   (b) If X is an ordered set, $E \subset X$ and there exists $y \in E$ such that $x \le y$ for all $x \in E$, then $y = \sup E$. (In other words, if a set contains an upper bound of the set, it has to be the supremum.)

   (c) Given $x, y \in \mathbb{R}$, show that

   $$x + y = \sup\{r + s \,:\, r, s \in \mathbb{Q},\, r \le x, s \le y\}.$$

2. **Several 'norms' in $\mathbb{R}^k$.** For a vector $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we define the non-negative quantities

   $$|\mathbf{x}|_1 := \sum_{i=1}^k |x_i|, \qquad |\mathbf{x}|_2 := |\mathbf{x}| = \sqrt{\sum_{i=1}^k |x_i|^2}, \qquad |\mathbf{x}|_\infty := \max_{1\le i\le k} |x_i|.$$

   (a) Show that for $\ell \in \{1, \infty\}$ we have the properties

   $$\begin{aligned} |\mathbf{x}|_\ell = 0 \qquad &\Longleftrightarrow \qquad \mathbf{x} = \mathbf{0} = (0, \ldots, 0), \\ |\lambda\,\mathbf{x}|_\ell = |\lambda|\,|\mathbf{x}|_\ell \qquad &\forall \mathbf{x} \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}, \\ |\mathbf{x} + \mathbf{y}|_\ell \le |\mathbf{x}|_\ell + |\mathbf{y}|_\ell \qquad &\forall, \mathbf{y} \in \mathbb{R}^k. \end{aligned}$$

   (b) Find the constants $C_{\ell,m}(k)$ for $\ell, m \in \{1, 2, \infty\}$ and any $k \ge 1$ that make the inequalities

   $$|\mathbf{x}|_\ell \le C_{\ell,m}(k)|\mathbf{x}|_m \qquad \forall \mathbf{x} \in \mathbb{R}^k$$

   sharp. (By this we mean that the inequalities hold and you can find one vector for which the inequality is an equality.)

3. **From norms to metrics.** Let $\|\cdot\| : \mathbb{R}^k \to [0,\infty)$ be a function satisfying

$$\begin{aligned} \|\mathbf{x}\| = 0 \qquad & \iff \qquad \mathbf{x} = \mathbf{0} = (0,\ldots,0), \\ \|\lambda\,\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\| \qquad & \forall \mathbf{x} \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}, \\ \|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| \qquad & \forall, \mathbf{y} \in \mathbb{R}^k. \end{aligned}$$

Show that $d(\mathbf{x},\mathbf{y}_) := \|\mathbf{x}-\mathbf{y}\|$ defines a metric in $\mathbb{R}^k$.

4. **From metrics to norms.** Assume that $d_\star$ is a metric defined in $\mathbb{R}^k$ and that $d_\star$ is translation invariant

$$d_\star(\mathbf{x},\mathbf{y}) = d_\star(\mathbf{x}+\mathbf{z},\mathbf{y}+\mathbf{z}) \qquad \forall \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^k$$

and scale invariant

$$d_\star(\lambda\mathbf{x},\lambda\mathbf{y}) = |\lambda| d_\star(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}.$$

Show that $\|\mathbf{x}\| := d(\mathbf{x},\mathbf{0})$ defines a norm in $\mathbb{R}^k$, that is, this function satisfies the properties of Problem 2 above.

5. Let $d_0 : X \times X \to \mathbb{R}$ be a metric in the set X and consider the binary functions

$$d_1(x,y) := \min\{d_0(x,y),1\} \qquad d_2(x,y) := \frac{d_0(x,y)}{1+d_0(x,y)} \qquad d_3(x,y) = c\,d_0(x,y)$$

(where in the last case $c > 0$ is a fixed number).

   (a) Show that d_1, d_2, and d_3 are metrics in X.
   (b) Show that for every $p \in X$, $r > 0$, and $i \in \{0,1,2,3\}$, there exists $r' > 0$ such that

   $$N^{d_j}_{r'}(p) \subset N^{d_i}_r(p) \qquad j \ne i.$$

   (Here $N^{d_i}_r(p)$ is the neighborhood about p with radius r with respect to the metric d_i.)
   (c) Prove that if a set $E \subset X$ is open with respect to the metric d_i, the it is open with respect to the three other metrics. In other words, the metric spaces (X,d_i) $(i = 0,\ldots,3)$ have the same open sets.

6. Let $d_1, d_2 : X \times X \to \mathbb{R}$ be two different metrics in X. Show that

$$d(x,y) := d_1(x,y) + d_2(x,y)$$

is a metric in X.

7. Give a rigorous proof that the sets

$$(a,b] := \{x \in \mathbb{R} \,:\, a < x \le b\}, \qquad [a,b) := \{x \in \mathbb{R} \,:\, a \le x < b\}$$

are not open (in $\mathbb{R}$ with the distance defined by the absolute value of the difference). Show that the interior of both sets is (a,b).

8. Let X be any set and consider the discrete distance

$$d(p,q) := \begin{cases} 1, & p \neq q, \\ 0, & p = q. \end{cases}$$

   Describe all neighborhoods of a point. Show that every subset of X is open.

9. Let X be a metric space where for every $p \in X$ the singleton set $\{p\}$ is open. Show that every subset of X is open.

10. (R2.9) (a)-(c)

11. (R2.11)