Problems for Chapter 2 (Part II)

by F.J.Sayas, for MATH 600

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Do not share these lists of problems outside the scope of the course. Problems listed as (R2.X) correspond to Rudin, Chapter 2, Problem X.

1. **Examples of convex sets.** We say that a subset C of $\mathbb{R}^k$ is convex, when for every $\mathbf{x}, \mathbf{y} \in C$, the segment $[\mathbf{x}, \mathbf{y}] = \{(1-t)\mathbf{x} + t\,\mathbf{y} \,:\, 0 \le t \le 1\}$ is a subset of C. Show that the following sets are convex in $\mathbb{R}^k$:
   (a) $\mathbb{R}^k$ itself.
   (b) Any open or closed ball.
   (c) Any closed k-cell $\{\mathbf{x} \in \mathbb{R}^k \;:\; a_i \le x_i \le b_i \quad \forall i\}$, where $a_i < b_i$ are given numbers.
2. Let X be a metric space such that for all $x \in X$ and $r > 0$ the neighborhood $N_r(x)$ contains infinitely many points. Show that $E^\circ \subset E'$, that is, every interior point of $E \subset X$ is a limit point of E.
3. Let X be a metric space and $E \subset X$. Show that:
   (a) If $y \in X$ and $c > 0$ satisfy that $d(x, y) \ge c$ for all $x \in X$, then $y \notin E'$.
   (b) If A is open and $A \subset E^c$, then $\overline{E} \subset A^c$.
4. Find the interior points, limit points, and closure of the following subsets of $\mathbb{R}$ with the Euclidean metric. Say if they are open or closed.
   (a) $\mathbb{Z}$.
   (b) $\mathbb{Q}$.
   (c) $(1, 2) \cup (2, 3)$.
   (d) $\{1/n^2 \,:\, n \in \mathbb{N}\} \cup \{0\}$.
5. Let X be a metric space, $x \in X$ and $r > 0$.
   (a) Show that
   $$\{y \in X \,:\, d(x, y) \le r\} \qquad \text{and} \qquad \{y \in X \,:\, d(y, x) = r\}$$
   are closed.

(b) Show that

$$\overline{N_r(x)} \subset \{y \in X \,:\, d(x, y) \leq r\}.$$

(c) Give an example of a case where the above is not an equality.

(d) Show that in $\mathbb{R}^k$ with the Euclidean distance

$$\overline{N_r(x)} = \{y \in X \,:\, d(x, y) \leq r\}.$$

6. **The boundary of a set.** Let X be a metric space and $E \subset X$. We define the boundary of E by

$$\partial E := \overline{E} \cap \overline{E^c}.$$

(a) Show that ∂E is closed and $\partial E = \partial E^c$.

(b) Show that $x \in \partial E$ if and only if

$$\forall r > 0 \quad N_r(x) \cap E \neq \emptyset \quad \text{and} \quad N_r(x) \cap E^c \neq \emptyset.$$

(c) Show that $\partial E = \overline{E} \setminus E^\circ$. (This gives an equivalent definition for the boundary.)

(d) Show that E is closed if and only if $\partial E \subset E$.

(e) When $X = \mathbb{R}^k$ with the Euclidean distance, find the boundary of

$$\{\mathbf{x} \in \mathbb{R}^k \,:\, |\mathbf{x} - \mathbf{x}_0| < r\} \quad \text{and} \quad \{\mathbf{x} \in \mathbb{R}^k \,:\, |\mathbf{x} - \mathbf{x}_0| \leq r\},$$

where $\mathbf{x}_0 \in \mathbb{R}^k$ and $r > 0$.

7. **Different metrics, same open sets.** Let X be a metric space with a metric d. Let us consider the metrics

$$d_1(x, y) := \min\{d(x, y), 1\}, \qquad d_2(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

Show that A is open in X with respect to the metric defined by d if and only if it is open with respect to the metric defined by d_1. Repeat the proof for d_2. (Hint. For some computations it might help to realize that x is interior to a set E if and only if there exists $r < 1$ such that $N_r(x) \subset E$.)

8. (R2.6)

9. (R2.7)

10. (R2.8)

11. (R2.12)