Problems for Chapter 2 (Part III)

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Do not share these lists of problems outside the scope of the course. Problems listed as (R2.X) correspond to Rudin, Chapter 2, Problem X.

1. **Bounded sets in metric spaces.** In a metric space X we say that $E \subset X$ is bounded when there exist $p \in X$ and $r > 0$ such that

$$d(p, q) \le r \qquad \forall q \in E.$$

   Prove that:

   (a) E is bounded if an only if there exists $p \in X$ such that

$$\{d(p, q) \,:\, q \in E\} \subset \mathbb{R}$$

   is upper bounded in $\mathbb{R}$.

   (b) E is bounded if and only if for all $p \in X$ there exists $r_p > 0$ such that

$$d(p, q) \le r_p \qquad \forall q \in E.$$

   (c) E is bounded if and only if E is contained in an open neighborhood.

   (d) Every compact set in X is bounded.

2. Let X be any infinite set endowed with the discrete metric. Show that every $E \subset X$ is closed and bounded, but only finite subsets of X are compact.

3. **Compact sets are totally bounded.** Let $E \subset X$ be a compact set in a metric space. Show that

$$\forall \varepsilon > 0 \quad \exists x_1, \ldots, x_n \in E \text{ such that } E \subset \bigcup_{j=1}^{n} N_\varepsilon(x_j).$$

   (If a set E satisfies this property it is called totally bounded. We will see in the next chapter that totally bounded closed sets in a complete metric space are compact.)

4. Give an example of a countable bounded set in $\mathbb{R}$ that is not compact.

5. Give an example of F_n closed such that $\cup_n F_n$ is not closed.
6. (R2.12)
7. (R2.13)
8. **Separable metric spaces.** A metric space X is separable if it contains $E \subset X$ such that E is countable and $\overline{E} = X$.
   (a) Show that $\mathbb{R}^k$ is separable. (Consider $E = \mathbb{Q}^k$.)
   (b) If an infinite set X is endowed with the discrete metric, show that X is separable if and only if X is countable.
   (c) (R2.23)
   (d) (R2.24)
   (e) (R2.25)