Problems for Chapter 3 (Part I)

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Problems listed as (R3.X) correspond to Rudin, Chapter 3, Problem X. For the first group of problems we need some definitions for subsets of a metric space X:

- A set $E \subset X$ is totally bounded if
$$\forall \varepsilon > 0 \quad \exists x_1, \ldots, x_n \in E \text{ such that } E \subset \bigcup_{j=1}^{n} N_\varepsilon(x_j).$$
- A set $E \subset X$ is complete if every Cauchy sequence in E is convergent to an element of E.
- A set $E \subset X$ is sequentially compact when every sequence on E contains a subsequence that converges to an element of E.

We have already seen that compact sets are totally bounded.

1. Show that a Cauchy sequence that contains a convergent subsequence is convergent (to the same limit as the subsequence).
2. **Characterization of closed sets with sequences.** Show that $E \subset X$ is closed if and only if convergent sequences of elements of E have their limits in E.
3. **Complete sets.** Show the following:
   (a) A closed subset of a complete space is complete.
   (b) A complete set is closed.
   (c) A sequentially compact set is complete.
4. **Equivalences.** In this exercise we are going to show that

   compact
   sequentially compact
   totally bounded and complete

   are equivalent concepts.

(a) Show that every compact set is sequentially compact. (Hint. Counterpositive. Start with a sequence with no convergent subsequences. For every x find $N_{\varepsilon_x}(x)$ which contains only finitely many elements of the sequence.)

(b) Show that every sequentially compact set is totally bounded. (Hint. Counterpositive. Build a sequence satisfying $d(x_n, x_m) \geq \varepsilon$ for all n, m.)

(c) Let $\{G_\alpha : \alpha \in I\}$ be an open cover of a sequentially compact set E. We want to prove the following statement:

$\exists \varepsilon > 0$ such that $\forall x \in E$, $\exists \alpha$ with $N_\varepsilon(x) \subset G_\alpha$.

We will do this by contradiction. Fill up the details of the following argument. If the property does not hold we can build a sequence such that $N_{1/n}(x_n)$ is not contained in any G_α. Let $x^\star$ be a subsequential limite of that sequence. We can find $N \in \mathbb{N}$ and α such that $N_{1/N}(x^\star) \subset G_\alpha$ and then take M large enough (use the corresponding subsequence) such that

$$N_{1/M}(x_M) \subset N_{1/m}(x^\star).$$

This leads to a contradiction.

(d) Show that every sequentially compact set is compact. (Hint. Use that sequentially compact sets are totally bounded.)

(e) Show that if a set is totally bounded, every sequence $\{x_n\}$ in the set contains a Cauchy subsequence. This is done using a Cantor diagonal argument. Let $\{x_{n,0}\}$ be a sequence (w.l.o.g. you can assume all its elements are different). By induction, find p_ℓ and $\{x_{n,\ell}\}$, subsequence of $\{x_{n,\ell-1}\}$, such that

$$x_{n,\ell} \in B_{1/2^\ell}(p_\ell) \qquad \forall n.$$

(Note that instead of writing $x_{n_k,\ell-1}$ for the subsequence, we write $x_{k,\ell} = x_{n_k,\ell-1}$.) The diagonal sequence, $y_n := x_{n,n}$, is a subsequence of the original sequence and is Cauchy.

Wrap up the entire collection of results above to show that compactness is equivalent to sequential compactness and to complete total boundedness.

5. **Three problems on numerical sequences.**
   - (R3.1)
   - (R3.2)
   - (R3.3)
6. (R3.23)
7. (R3.24) — This problem gives a detailed definition of the completion of a metric space. It is long and demanding. Give a serious try at least to the first questions.
8. (R3.25)