Problems for Chapter 3 (Part II)

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Do not share these lists of problems outside the scope of the course. Problems listed as (R3.X) correspond to Rudin, Chapter 3, Problem X.

1. **Review. Relatively compact sets.** Let X be a metric space. Recall that a set $E \subset X$ is said to be bounded when $E \subset N_r(x)$ for some $x \in X$ and $r > 0$. We say that a set is **relatively compact** when its closure is compact.

   (a) Show that relatively compact sets are bounded. (Hint. Show that the closure of a bounded set is bounded.)

   (b) Show that subsets of relatively compact sets are relatively compact.

   (c) Show that in $\mathbb{R}^k$, relatively compact sets are equal to bounded sets.

   (d) Characterize bounded and relatively compact sets in X endowed with the discrete metric.

2. **Review. An exercise on closures.** Show that

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \qquad \text{and} \qquad \overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

   Give an example where $\overline{A \cap B}$ is a proper subset of $\overline{A} \cap \overline{B}$. Can these relations be extended to arbitrary unions and intersections?

3. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ with the property $x_{n+1} \geq x_n$ for $n \geq N$. Show that if a subsequence converges, then the sequence converges. Show that if a subsequence diverges to $+\infty$, then the entire sequence diverges to $+\infty$.

4. Let $\{x_n\}$ be a sequence in a metric space X. Show that if $\lim x_{2n} = \lim x_{2n+1}$, then the sequence converges.

5. Show that the discrete measure defines a complete metric space on any set.

6. Show that $x_n \to x$ (in a metric space with metric d) implies

$$d(x_n, y) \to d(x, y) \qquad \forall y \in X.$$

7. **Two equivalent metrics on the same set.** Let X be a metric space with metric d and let

$$\check{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

define another metric in X. Show that convergent sequences with respect to both metrics are the same. Show that X is complete with respect to the metric d if and only if X is complete with respect to the metric $\check{d}$.

8. **Metrics defined with sequences of metrics.** Let $d_j : X \times X \to [0,\infty)$ be a metric for $j \geq 1$, and define

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_j(x,y)}{1+d_j(x,y)} = \lim_{J\to\infty} \sum_{j=1}^{J} \frac{1}{2^j} \frac{d_j(x,y)}{1+d_j(x,y)}.$$

   (a) Show that $d(x,y)$ is well defined and $d(x,y) \leq 1$ for all x, y. (Hint. It is defined as the limit of a non-decreasing sequence.)

   (b) Show that d is a metric in X.

   (c) Show that $x_n \to x$ with respect to the metric d if and only if $x_n \to x$ with respect to the metric d_j for all j. (Hint. For the difficult implication, given $\varepsilon > 0$ take J large enough so that $1/2^{J+1} < \varepsilon$.)