Problems for Chapter 4 and Prelim Exam problems

by F.J.Sayas, for MATH 600

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In all the following exercises X is a general metric space.

1. **A problem about sequences.** Knowing that the sequence

$$x_n = \sum_{j=0}^{n} \frac{1}{j!}$$

is monotonically increasing and bounded, we know that its limit exists. We call this limit

$$e = \lim_n x_n = \sum_{j=0}^{\infty} \frac{1}{j!}.$$

Our goal is to show that

$$e = \lim_n \left(1 + \frac{1}{n}\right)^n.$$

(a) Show that

$$y_n := \left(1 + \frac{1}{n}\right)^n = 1 + \sum_{j=1}^{n} \frac{1}{j!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{j-1}{n}\right) \le x_n \qquad \forall n$$

and therefore $\limsup y_n \le e$.

(b) For fixed m, show that

$$y_n \ge 2 + \sum_{j=2}^{m} \frac{1}{j!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{j-1}{n}\right) \qquad \forall n \ge m$$

and therefore

$$\liminf y_n \ge x_m.$$

Use this to show that $\liminf y_n \ge e$ and to prove the main result.

2. Consider the spaces $X = [0, 2\pi)$, with the Euclidean metric, $Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$ also endowed with the Euclidean metric, and the function $f(t) = (\cos t, \sin t)$.

   (a) Show that X is not a compact metric space and Y is a compact metric space.

   (b) Assuming that we know that the function $\sin : \mathbb{R} \to \mathbb{R}$ is continuous, show that f is continuous.

   (c) Show that f is bijective but the inverse function $g : Y \to X$ is discontinuous at $(1, 0)$.

3. Let $f : X \to \mathbb{C}$ be a continuous function (X is a metric space). Show that the set

$$\{x \in X \,:\, f(x) = 0\}$$

   is closed. (You should be able to find at least three different proofs of this result.)

4. Let $f : X \to \mathbb{R}$ be a continuous function. Show that if $f(x_0) > 0$, then there exists a neihborhood of x_0 where f is positive.

5. (R4.2)

6. (R4.4)

7. (R4.6)

8. (R4.14)

These problems are taken from past preliminary exams.

1. Let (X, d_X) and (Y, d_Y) be metric spaces. The Cartesian product $Z = X \times Y$ becomes a metric space if equipped with the distance

$$d_Z((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

   (a) Show that a sequence $\{(x_n, y_n)\}$ converges to (x, y) in Z if and only if $x_n \to x$ in X and $y_n \to y$ in Y.

   (b) Consider now a function $f : X \to Y$ and its graph

$$G = \{(x, f(x)) \,:\, x \in X\} \subset Z.$$

   Assume that Y is compact. Show that f is continuous if and only if G is closed in Z.

   (c) Show that the previous result does not hold when Y is not compact.

2. Let $\{x_n\}$ be a sequence of real numbers in the interval $[0, 3/2]$. Suppose that $\lim_{n\to\infty} |x_n - x_{n+1}| = 1$. Prove that no subsequence of $\{x_n\}$ converges to $3/4$.

3. Let $\{x_n\}$ be a sequence in C such that for every $\varepsilon > 0$ there exists a convergent sequence $\{y_n\}$ in $\mathbb{C}$ such that $\sup_n d(x_n, y_n) < \varepsilon$. Prove that $\{x_n\}$ is convergent.

4. Let X be a metric space and $A, B \subset X$.

   (a) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and find an example in $\mathbb{R}$ where the two sets are different.

   (b) Show that if A is open, then $A \cap \overline{B} \subset \overline{A \cap B}$. Show that the assumption that A is open cannot be removed.

   (c) Show that if A and B are dense in X and A is open, then $A \cap B$ is dense in X. Show that the assumption that A is open cannot be removed.

5. Let $f : (0, 1) \to \mathbb{R}$ be uniformly continuous. Show that f is bounded, i.e., there exists $M > 0$ such that $|f(x)| < M$ for all x. Show that $f(0_+)$ exists.

6. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and assume that $\lim_{x\to\infty} f(x)$ exists. Show that f is uniformly continuous.

7. Let ℓ^∞ be the set of bounded sequences of real numbers, that is $\mathbf{x} = \{x_n\}$ is a sequence such that there exists C with $|x_n| \leq C$ for all n. Consider the following function
$$d(\mathbf{x}, \mathbf{y}) = \sup_n |x_n - y_n|.$$
Show that it is a metric and that the set
$$\{\mathbf{x} \in \ell^\infty \, : \, d(\mathbf{x}, \mathbf{0}) \leq 1\}$$
is closed and bounded, but not compact. (Here $\mathbf{0}$ is the zero sequence.)