Problems for Chapter 5 and Prelim Exam problems

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In all the following exercises X is a general metric space. Problems 2-7 have appeared in past preliminary exams.

1. **Growth observed with derivatives.** Use the Mean Value Theorem to prove the following result. Let $f : (a, b) \to \mathbb{R}$ be differentiable in every point.
   (a) If $f'(x) \geq 0$ at every point, then f is monotocinally increasing.
   (b) If $f'(x) \leq 0$ at every point, then f is monotonically decreasing.
   (c) If $f'(x) =$ at every point, then f is constant.

2. Define $f : \mathbb{R} \to \mathbb{R}$ differentiable everywhere and satisfying that there exists $x_0 \in \mathbb{R}$ such that $f'(x_0) = 0$ but x_0 is not a local maximum or minimum.

3. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable in a neighborhood of x_0. Prove that
$$\lim_{h\to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

4. Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be twice differentiable. Denote
$$M_0 = \sup_{x\in(a,\infty)} |f(x)|, \qquad M_1 = \sup_{x\in(a,\infty)} |f'(x)|, \qquad M_2 - \sup_{x\in(a,\infty)} |f''(x)|.$$
   Prove that
$$M_1^2 \leq 4M_0M_2.$$
   (Hint. Use Taylor's theorem for $f(x + 2h)$ and $h > 0$.)

5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and differentiable in (a, b). Suppose that $f(a) = 0$ and there exists $K > 0$ such that $|f'(x)| \leq K\,|f(x)|$ for all $x \in (a, b)$. Prove that f is identically zero.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable in $\mathbb{R}$ and twice differentiable at a, with $f'(a) = 0$ and $f''(a) > 0$. Prove that f has a local minimum at a.

7. Suppose $f : [0,1] \to \mathbb{R}$ is continuous in $[0,1]$ and differentiable in $(0,1)$. If

$$\lim_{x \to 0} f'(x) = 3,$$

show that f has a one-sided derivative at $x = 0$ and $f'(0) = 3$.

8. (R5.1)

9. (R5.2)

10. (R5.3)

11. (R5.12)

12. (R5.17)

13. (R5.18)