Problems for Chapter 6 and Prelim Exam problems

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Problems 1-4 have appeared in past preliminary exams.

1. Suppose f is a positive continuous function on $[a,b]$ and M is the largest value of f on $[a,b]$. Prove that
$$\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{1/n} = M.$$
(Hint. Given $\varepsilon > 0$, prove that there is a positive length interval on which $f(x) \geq M - \varepsilon$.)

2. Let $x_0 \in (0,1)$ and $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 0, & x \neq x_0, \\ 1, & x = x_0. \end{cases}$$
Prove that f is integrable and compute the integral. (To make it more interesting, use only the definition of the Riemann integral.)

3. Suppose $g : [a,b] \to \mathbb{R}$ is bounded and g^2 is Riemann integragle. Does it follow that g is Riemann integrable?

4. (i) Show that if f is twice continuously differentiable and convex (concave up) in $[a,b]$, then
$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx.$$
(Hint. Conside the Taylor expansion around the midpoint of the interval.)

   (ii) Use the previous result to prove the estimate
$$\sum_{n=2}^{\infty}\frac{1}{1+n^2} \leq \frac{\pi}{2} - \arctan(3/2).$$

5. (R6.1)

6. (R6.4)
7. (R6.5)
8. (R6.7)
9. (R6.8)
10. (R6.9)