Problems for Chapter 7

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1. **Dini's Theorem.** Prove the following theorem: *Let f_n be an increasing sequence in $\mathcal{C}(K)$, where K is compact. If there exists $f \in \mathcal{C}(K)$ such that $f_n \to f$ pointwise, then the convergence is uniform.* Sketch of the proof: consider the functions
$$g_n = f - f_n = |f - f_n| \in \mathcal{C}(K).$$
Given $\varepsilon > 0$, consider the sets
$$K_n = \{x \in K \,:\, g_n(x) \geq \varepsilon\}.$$
Show that $K_N = \emptyset$ for some N.

2. **Convergence of power series.** Consider the power series
$$\sum_{n=1}^{\infty} a_n z^n,$$
for given complex coefficients $\{a_n\}$.
   (a) Show that if the series converges absolutely for a value z_0 and $r = |z_0|$, then the series converges uniformly in the closed disk $\{z \in \mathbb{C} \,:\, |z| \leq r\}$.
   (b) Show that power series converge uniformly on compact sets of their region of convergence $\{z \in \mathbb{C} \,:\, |z| < R\}$, where R is the radius of convergence of the series.

3. **Some problems on algebras of functions.** In all these cases $\mathcal{A}$ is a self-adjoint (i.e., closed by conjugation) algebra of functions $f : E \to \mathbb{C}$, with no particular structure on the set E. We define
$$\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} \,:\, f(x) \in \mathbb{R} \quad \forall x \in E\}.$$
   (a) Show that for all $f \in \mathcal{A}$, the functions $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are also in $\mathcal{A}$.
   (b) Show that
$$\mathcal{A}_{\mathbb{R}} = \{\mathrm{Re}\, f \,:\, f \in \mathcal{A}\} = \{\mathrm{Im}\, f \,:\, f \in \mathcal{A}\}.$$
   (c) Show that $\mathcal{A}_{\mathbb{R}}$ is a real algebra (with real scalars, instead of complex scalars).

(d) Show that if $\mathcal{A}$ separates points, so does $\mathcal{A}_{\mathbb{R}}$ (Hint. Given $x_1 \neq x_2$ and a function $f \in \mathcal{A}$ that separates x_1 and x_2, you can choose the real or the imaginary part.)

(e) Show that if $\mathcal{A}$ vanishes at no point, the neither does $\mathcal{A}_{\mathbb{R}}$.

4. Let $\mathcal{A} \subset \mathcal{C}(K)$ be an algebra, where K is a compact metric space. Show that $\overline{\mathcal{A}}$ is an algebra. (Hint. Characterize elements of the closure as limits of sequences.)

5. **Simple consequences of Weierstrass's Theorem.** Assume that we have proved that: *for every continuous* $f : [0, 1] \to \mathbb{C}$ *there exists a sequence of polynomials* $\{p_n\}$ *such that* $p_n \to f$ *uniformly.* Prove the following:

   (a) For every continuous $f : [a, b] \to \mathbb{C}$, there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly in $[a, b]$. (Hint. Change variables.)

   (b) For every continuous $f : [a, b] \to \mathbb{R}$, there exists a sequence of polynomials $\{p_n\}$, with real coefficients, such that $p_n \to f$ uniformly in $[a, b]$. (Hint. Take real parts.)

   (c) For every continuous $f : [a, b] \to \mathbb{C}$ and fixed $c \in [a, b]$, there exists a sequence of polynomials $\{p_n\}$ with $p_n(c) = f(c)$ such that $p_n \to f$ uniformly in $[a, b]$. (Hint. If $q_n \to f$ uniformly, then $q_n(c) \to f(c)$.)

6. (R7.1)

7. (R7.2)

8. (R7.3)

9. (R7.9)

10. (R7.16)

11. (R7.18)