Connectedness

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In this worksheet, X will be a general metric space, unless something else is explicitly stated. When we refer to $\mathbb{R}$ or $\mathbb{R}^k$, we will assume that the Euclidean metric is considered. Prove all the numbered assertions.

Let A and B be subsets of a metric space X. We say that A and B are **separated** when

$$A \cap \overline{B} = \emptyset \qquad \text{and} \qquad \overline{A} \cap B = \emptyset.$$

Note that by definition, two separated sets have to be disjoint.

(1) The sets $(0,1)$ and $(1,2)$ are separated in $\mathbb{R}$. However, $(0,1]$ and $(1,2)$ are not.

(2) If we consider the discrete metric, disjoint is equivalent to separated.

(3) If A and B are closed and disjoint, they are separated.

(4) If A and B are open and disjoint, they are separated. (Hint. If they are not separated, we can assume without loss of generality that we can find $x \in A'$, $x \notin A$, $x \in B$. Take neighborhoods now.)

(5) If $A_1 \subset A_2$ and $B_1 \subset B_2$ and the sets A_2 and B_2 are separated, so are A_1 and B_1.

We say that $E \subset X$ is **connected** when

$$\left.\begin{array}{c} E = A \cup B \\ A \text{ and } B \text{ are separated} \end{array}\right\} \qquad \Longrightarrow \qquad A = \emptyset \quad \text{or} \quad B = \emptyset.$$

(6) $\mathbb{Z}$ is not connected in $\mathbb{R}$.

(7) $(0,2) \setminus \{1\} = (0,1) \cup (1,2)$ is not connected in $\mathbb{R}$.

Proving that a given set is connected is somewhat more difficult. We are going to examine the case of $\mathbb{R}$ in detail.

(8) (A side result.) If $A \subset \mathbb{R}$ is upper bounded, then $\sup A \in \overline{A}$. (Hint. If $y = \sup A \notin A$ (otherwise, there is nothing to prove), then $(y - r, y) \cap A \neq \emptyset$ for all r.)

(9) Let $x, y \in E$ satisfy $x < y$ and let $z \in (x, y)$. If $z \notin E$, then

$$E = ((-\infty, z) \cap E) \cup ((z, \infty) \cap E)$$

is a partition of E into separated sets so E is not connected.

(10) If E is not connected, we can take $x < y$ where $x \in A$, $y \in B$ and $E = A \cup B$ is a partition of E into separated sets. Let

$$z = \sup(A \cap [x, y]).$$

(a) $z \in \overline{A}$ and therefore $z \in E$ if and only if $z \in A$.

(b) If $z \in A$, there exists $t \in (z, y)$ such that $t \notin E$.

Therefore, if E is not connected, we can find $x < y$ in E such that $(x, y) \not\subset E$.

(11) $E \subset \mathbb{R}$ is connected if and only if

$$x, y \in E \quad x < y \qquad \Longrightarrow \qquad (x, y) \subset E.$$

(12) $E \subset \mathbb{R}$ is connected if and only if E is an interval. (Hint. Examine bounded and unbounded cases.)

(13) A set E in $\mathbb{R}^k$ is convex when for all $\mathbf{x}, \mathbf{y} \in E$, the segment $\{(1-t)\mathbf{x}+t\,\mathbf{y} \,:\, t \in (0, 1)\}$ is a subset of E. Convex subsets of $\mathbb{R}^k$ are connected. (Hint. This is Problem 21 of Chapter 2 of Rudin. The problem contains several intermediate steps.)