## Countable sets

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September 12, 2018

In some of these exercises you can write clear while intuitive (not notationally charged) proofs. Note that Rudin's notation for the set of positive integers is $J$. We will stick to the more standard notation $\mathbb{N}$.

Let us recall that a bijection between two sets $X$ and $Y$ is a map $\phi: X \rightarrow Y$ such that

$$
\forall y \in Y \quad \exists!x \in X \quad \text { satisfying } \quad \phi(x)=y
$$

Therefore, we can define another bijection $\phi^{-1}: Y \rightarrow X$ given by $x=\phi^{-1}(y)$ being the unique $x \in X$ such that $\phi(x)=y$. This justifies talking about bijections between the sets $X$ and $Y$.
A set $X$ is countable if there exists a bijection between the set of natural numbers $\mathbb{N}$ and $X$. If this bijection is denoted $\phi: \mathbb{N} \rightarrow X$, we can write $x_{n}=\phi(n)$ for every $n \in \mathbb{N}$ and it happens that $X=\left\{x_{n}: n \in \mathbb{N}\right\}$, that is, we can tag the elements of $X$ with an integer index. (Do not confuse this notation with the notation for sequences. In sequences elements can be repeated, while in sets elements are always different.) Building the bijection from $\mathbb{N}$ to a countable set $A$ is informally referred as counting the set $A$.
(1) Show that $\mathbb{N}$ is countable.
(2) Show that the set of even positive numbers is countable.
(3) Show that the set of odd positive numbers is countable.
(4) Let $A \subset \mathbb{N}$ and assume $A$ is not finite. Show that we can list all the elements of $A$ as

$$
n_{1}<n_{2}<\ldots<n_{k}<\ldots
$$

Use this to prove that $A$ is countable. (Build the bijection between $\mathbb{N}$ and $A$.)
(5) Show that every subset of a countable set is either finite or countable. (Hint. Use the inverse bijection from the total set to identify a subset of $\mathbb{N}$.)
(6) Let $A$ and $B$ be disjoint subsets of $X$, with $A$ finite and $B$ countable. Show that $A \cup B$ is countable. (Hint. First count all of $A$.)
(7) Let $A$ and $B$ be disjoint countable subsets of $X$. Show that $A \cup B$ is countable. (Hint. Count elements of $A$ and $B$ alternatively, one from each set at a time. Or count elements of $A$ with even numbers and elements of $B$ with odd numbers.) Show that the finite disjoint union of countable sets $A_{1} \cup A_{2} \cup \ldots \cup A_{N}$ is countable.
(8) Show that $\mathbb{Z}$ is countable.
(9) Show that the set

$$
\mathbb{N} \times \mathbb{N}:=\{(n, m): n, m \in \mathbb{N}\}
$$

is countable. (Hint. Count elements diagonally, by considering all elements ( $n, m$ ) with $n+m=k$ for increasing $k$.)
(10) Show that if $A$ and $B$ are countable, then $A \times B$ is countable. Show that the countable union of countable sets is countable.
(11) Show that $\mathbb{Q}$ is countable. (Hint. The set of positive rational numbers can be identified with a subset of $\mathbb{N} \times \mathbb{N}$. How?)

You might now wonder whether there are uncountable sets and whether real numbers are countable. Here is a really clever argument due to Georg Cantor. Consider the set of all possible sequences with elements in the binary set $\{0,1\}$ :

$$
\begin{aligned}
S=\{0,1\}^{\mathbb{N}} & :=\left\{\mathbf{s}=\left(s_{1}, s_{2}, s_{3}, \ldots\right): s_{n} \in\{0,1\} \quad \forall n\right\} \\
& \equiv\{\phi:\{0,1\} \rightarrow \mathbb{N}: \phi \text { is a map }\} .
\end{aligned}
$$

Assume that $S$ is countable so that we can list all the elements of $S$ as

$$
\mathbf{s}_{n}=\left(s_{n, 1}, s_{n, 2}, s_{n, 3}, \ldots\right) \quad n \in \mathbb{N} .
$$

Consider now the sequence

$$
\mathbf{t}=\left(t_{1}, t_{2}, t_{3}, \ldots\right) \quad \text { given by } \quad t_{j}= \begin{cases}1, & \text { if } s_{j, j}=0 \\ 0, & \text { if } s_{j, j}=1\end{cases}
$$

It is clear that $\mathbf{t} \neq \mathbf{s}_{n}$ for all $n$ since $t_{n} \neq s_{n, n}$. This leads to a contradiction with the fact that we have counted all the elements of $S$.
(12) Given $\mathbf{s}=\left(s_{1}, s_{2}, s_{3}, \ldots\right) \in\{0,1\}^{\mathbb{N}}$, consider

$$
x=s_{1} 10^{-1}+s_{2} 10^{-2}+s_{3} 10^{-3}+\ldots,
$$

that is $x$ is the real number whose decimal digits are either 0s or 1 s and are given by the elements of the sequence $\mathbf{s}$. Use this to show that

$$
[0,1]=\{x \in \mathbb{R}: 0 \leq x \leq 1\}
$$

is not countable. (Hint. It contains an uncountable subset.)
(14) Prove that $\mathbb{R}$ is not countable.

