NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2018 | Midterm exam # 1 | October 10

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.** If you think a question might be incorrect, write it in your exam and move on. Nothing will be modified during the exam.

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) Signature Date

Write here a three digit number

Write the same number

1	20 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	20 pts	
TOTAL	100 pts	

1. Consider the set

$$E = (-1, 1) \times \{0\} = \{(t, 0) \,:\, -1 < t < 1\} \subset \mathbb{R}^2,$$

where we use the Euclidean metric in $\mathbb{R}^2$.

(a) Show what E°, $\overline{E}$, and E' are. All assertions need to be justified.

(b) Is E open/closed/compact/bounded/countable?

2. Let X be a metric space and $E \subset X$.

   (a) Let $p \in E$. Show that p is isolated if and only if there exists $r > 0$ such that $N_r(p) \cap E$ is finite.

   (b) Show that $p \in E'$ if and only if for all $r > 0$, $N_r(p) \cap E$ is inifinite.

3. Let A_n for $n \in \mathbb{N}$ be subsets of a metric space.

(a) Show that

$$\bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n}.$$

(b) Find an example in $\mathbb{R}$ with the Euclidean metric where the inclusion is strict. (Hint. You can use sets $A_n = \{q_n\}$ with $q_n \in \mathbb{Q}$.)

4. In $\mathbb{R}^k$ with the Euclidean metric show that

$$\overline{A} \text{ is compact} \qquad \Longleftrightarrow \qquad A \text{ is bounded.}$$

(Hint. It might help to *prove* that for any $p \in \mathbb{R}^k$ and $r > 0$, the set $\overline{N_r(p)}$ is bounded.)

5. In a general metric space X show that $E \subset X$ is open if and only if

$$E = \bigcup_{\alpha \in \mathcal{A}} N_{r_\alpha}(p_\alpha),$$

where $\mathcal{A}$ is an index set (not necessarily countable).

6. In the metric space $X = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 > 0\}$, equipped with the Euclidean metric, we consider the set

$$E = \{\mathbf{x} \in X \, : \, |\mathbf{x}| \leq 1\},$$

which is clearly bounded.

   (a) Show that E is closed.

(b) Use the sets

$$G_n = \{\mathbf{x} \in X \,:\, |\mathbf{x}| < 2, \quad x_1 > 1/n\},$$

to prove that E is not compact.