NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2018 | Midterm exam # 2 | November 7

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.** If you think a question might be incorrect, write it in your exam and move on. Nothing will be modified during the exam.

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) | Signature | Date

Write here a three digit number

Write the same number

1	10 pts	
2	15 pts	
3	20 pts	
4	10 pts	
5	15 pts	
6	15 pts	
7	15 pts	
TOTAL	100 pts	

1. We say that E is dense in X (here X is a general metric space), when $\overline{E} = X$. Show that E is dense in X if and only if for every non-empty open set $G \subset X$, we have $E \cap G \neq \emptyset$.

2. Let A and B be subsets of a metric space X. Show that if A is open, then

$$A \cap \overline{B} \subset \overline{A \cap B}.$$

Give an example showing that the hypothesis on A cannot be removed.

3. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and assume that $\lim_{x\to\infty} f(x)$ exists. Show that f is uniformly continuous.

4. Show that $x_n \to x$ if and only if the following holds:

Every subsequence of $\{x_n\}$ contains a subsequence converging to x.

5. Let $f : (a, b) \to \mathbb{R}$ be continuous and strictly increasing.

   (a) Show that for any $a < x_1 < x_2 < b$, f defines a bijection between $[x_1, x_2]$ and $[f(x_1), f(x_2)]$.

(b) If $Y = f((a, b))$, show that $f^{-1} : Y \to (a, b)$ is continuous.

6. Let $f : X \to Y$ be a continuous function between two metric spaces, and let $H \subset Y$ be a finite set. Show that

$$\{x \in X \, : \, f(x) \in H\}$$

is closed in X. Give an example showing that the result does not hold for countable H.

7. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Knowing that $\sin : \mathbb{R} \to \mathbb{R}$ is differentiable and that $\sin(x)' = \cos(x)$, prove that f is differentiable at every point. (Hint. This is a little bit less obvious that you might think.)

(b) Show that f' has a discontinuity at $x = 0$ but that this discontinuity is neither removable nor of the first kind (jump).