Two exam problems

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1. Let X be a compact metric space and let $g_n : X \to \mathbb{R}$ be continuous functions satisfying
$$g_n(x) \geq g_{n+1}(x) \quad \forall x \in X, \forall n,$$
and
$$\lim_{n\to\infty} g_n(x) = 0 \quad \forall x \in X.$$

   (a) For arbitrary $\varepsilon > 0$, consider the sets
$$K_n^\varepsilon = \{x \in X \,:\, g_n(x) \geq \varepsilon\}.$$
   Show that
$$\bigcap_{n=1}^\infty K_n^\varepsilon = \emptyset$$
   and therefore there exists N such that $K_n^\varepsilon = \emptyset$ for all $n \geq N$.

   (b) Use (a) to prove that $g_n \to 0$ uniformly in X.

2. Consider the functions $f_n : [0,1] \to \mathbb{R}$ given by
$$f_n(x) = \frac{1+nx^2}{(1+x)^n}.$$

   (a) Compute
$$f(x) = \lim_{n\to\infty} f_n(x) \qquad x \in [0,1].$$

   (b) Show that $f_n \to f$ uniformly in $[\varepsilon, 1]$ for all $\varepsilon > 0$ but that the limit is not uniform in $[0,1]$.

   (c) Without computing the integrals on the left hand side, prove that
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$
   (Warning. This exercise can be solved with more advanced theorems of Lebesgue integration. Here you are asked to use only tools of Riemann integration.)