NAME:

MATH 600: Fundamentals of Real Analysis

Fall 2018 | Final exam | December 13

READ CAREFULLY THESE INSTRUCTIONS

- Your answers have to be written in a fully coherent and readable way. You can work on scrap paper, but only what is written in this document will be graded.
- Correct answers written in a careless and mathematically non-rigorous way will get practically no credit.
- Jusfity your answers explaining what arguments and results you are using to get to your conclusions.
- Do not write anything in the exam (except in this cover page) that reveals your identity.
- **You are not allowed to ask questions during the exam.** If you think a question might be incorrect, write it in your exam and move on. Nothing will be modified during the exam.

Sign and print your name and date here to show that you have read and understood these instructions.

Name (print) Signature Date

Write here a three digit number

Write the same number

1	15 pts	
2	10 pts	
3	10 pts	
4	15 pts	
5	20 pts	
6	10 pts	
7	20 pts	
TOTAL	100 pts	

1. Let K be a non-finite compact subset of a metric space. Show that there exists $E \subset K$ such that E is countable and $\overline{E} = K$.

2. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, non-negative and such that

$$\int_a^b f(x)^2 dx = 0.$$

Show that

$$\int_a^b f(x) dx = 0.$$

(Hint. Prove and use the inequality $a \leq \frac{1}{4\varepsilon}a^2 + \varepsilon$, valid for any a and $\varepsilon > 0$.)

3. Let $\{a_n\}$ be a non-decreasing sequence of real numbers and $\{b_n\}$ be a non-increasing sequence of real numbers. Assume that $a_n \leq b_n$ for all n. Show that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = [\lim_n a_n, \lim_n b_n].$$

4. Let $E \subset X$, where X is a general metric space. Show that $F \subset E$ is closed in E if and only if there exists F^* closed in X such that $F^* \cap E = F$.

5. Let $\{f_n\}$ be a non-decreasing sequence of continuous functions $f_n : K \to \mathbb{R}$, where K is compact in $\mathbb{R}$. Assume that there exists $f : K \to \mathbb{R}$ continuous, such that $f_n \to f$ pointwise.

   (a) Consider the functions $g_n = f - f_n$ and the sets

$$K_n = \{x \in K \,:\, g_n(x) \geq \varepsilon\}.$$

   Show that $K_N = \emptyset$ for some N. (Hint. Be sure that you verify all the hypotheses if you want to use a particular theorem.)

   (b) Prove that $f_n \to f$ uniformly in K.

6. Let $\mathbf{f} : X \to \mathbb{R}^k$ be continuous on the metric space X and let $\mathbf{d}_1, \ldots, \mathbf{d}_m \in \mathbb{R}^k$ be given vectors. Show that

$$\{x \in X \,:\, \mathbf{f}(x) \cdot \mathbf{d}_\ell \leq 0 \quad \ell = 1, \ldots, m\}$$

is closed.

7. Let the functions $f_n; [0,1] \to [0,1]$ satisfy

$$\begin{aligned} f_n(0) &= 1 && \forall n, \\ f_n'(x) &\leq 0 && \forall n, \forall x \in [0,1], \\ \lim_{n\to\infty} f_n(x) &= 0 && \forall x \in (0,1]. \end{aligned}$$

(a) Show that $\{f_n\}$ is uniformly convergent in $[\delta, 1]$ for all $\delta > 0$.

(b) Show that $\{f_n\}$ is not uniformly convergent in $[0,1]$.

(c) Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$