Can you rewrite and negate statements?

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When we write 'blackboard style' mathematics, we tend to use a lot of symbols, but sometimes symbols are implied (see example 1 below, where the quantifier is assumed by the implication sign). Try to understand precisely what the following assertions tell and **negate them.** When you negate $\exists$, avoid $\nexists$ and use a universal quantifier instead.
Note that you should not write mathematics in the style of the following statements unless you are focusing in the contents and you are writing 'for yourself.' Except in logic and some related branches, quantifiers are not used as part of text formulas, but they are commonly used in displayed formulas.

Negate the following statements:

1. (Here $X \subset \mathbb{R}$.) $x \in X \Longrightarrow x^2 \in X$, which can also be written as
$$x^2 \in X \qquad \forall x \in X.$$
2. (Here $X \subset \mathbb{R}$.)
$$\exists a \in \mathbb{R} \text{ such that } \forall x \in X, x < a.$$
3. (Here $X \subset \mathbb{R}$.)
$$\forall \delta > 0 \quad \exists x_1, x_2 \in X, \qquad \text{such that} \qquad |x_1 - x_2| > \delta.$$
4. (Here X, Y and Z are subsets of $\mathbb{R}$.)
$$\forall x \in X \qquad \exists y \in Y \qquad \text{such that} \qquad xy < z \quad \forall z \in Z.$$
5. (Here $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given.)
$$\forall \varepsilon > 0, \qquad \exists \delta > 0 \quad \text{such that} \quad |x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon$$
6. (Here $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\ell \in \mathbb{R}$ are given.)
$$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \text{such that} \quad |x_n - \ell| < \varepsilon \quad \forall N \geq N_0.$$

7. (Here $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ are given.)

$$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \text{such that} \quad n, m \geq N_0 \quad \Longrightarrow \quad |x_n - x_m| < \varepsilon.$$

8. (Here $f : \mathbb{R} \to \mathbb{R}$ is given.)

$$\exists L > 0 \quad \text{such that} \quad |f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}.$$