Maps and the meaning of f^{-1}

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> We will use the symbol $\subset$ understanding that when $U \subset V$, we admit the possibility of $U = V$. Note that there are different standards for this and some people prefer differentiating the symbols $\subseteq$ (admitting equality) and $\subset$ (strict). When you are asked to build an example it is completely fine to draw some clear Venn diagrams describing the example. In all the following problems, you can find the requested examples with very small sets.

Let X and Y be two sets. A map (function, mapping, or application) is a relation $f : X \to Y$, that associates to **every** $x \in X$ a unique element in Y, which we denote $f(x)$. If $A \subset X$, we denote

$$f(A) := \{y \in Y \,:\, y = f(x) \text{ for some } x \in A\} = \{f(x) \,:\, x \in A\} \subset Y.$$

Note the abuse of notation, since we are using f applied to elements and to subsets as well.

(1) Show that if $A_1 \subset A_2 \subset X$, then $f(A_1) \subset f(A_2)$.

(2) Build an example where $A_1 \subset A_2$, $A_1 \neq A_2$ and yet $f(A_1) = f(A_2)$.

(3) Prove the following formulas

$$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2), \qquad f(A_1 \cup A_2) = f(A_1) \cup f(A_2).$$

In principle our map $f : X \to Y$ is **not bijective** (bijective means that for every $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$), so we do not have the right to define the **inverse map** f^{-1}. However, we can define the 'inverse map' acting on sets: if $B \subset Y$, we define

$$f^{-1}(B) = \{x \in X \,:\, f(x) \in B\} \subset X.$$

(4) Show that $f^{-1}(Y) = X$.

(5) Show that if $B_1 \subset B_2 \subset Y$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$.

(6) Show that $f^{-1}(B^c) = (f^{-1}(B))^c$, where the upperscript c denotes the complement.

An interesting question arises when we think about what happens if we first apply f and then f^{-1} or viceversa. Here is the answer to that question.

(7) Show that $A \subset f^{-1}(f(A))$.

(8) Show that $f(f^{-1}(B)) \subset B$.

(9) Give examples showing that the subset symbol in two statements above can be strict, i.e., construct $f : X \to Y$ and find $A \subset X$ and $B \subset Y$ such that $A \neq f^{-1}(f(A))$ and $f(f^{-1}(B)) \neq B$.