Series

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In this worksheet, sequences will be tagged with the positive integers or with non-negative integers, that is, we will consider a_n for $n \geq 0$.

Consider a sequence $\{a_n\}$ of complex numbers and the new sequence of **partial sums**

$$s_n = a_1 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

If $\{s_n\}$ converges, we say that the series

$$\sum_{n=1}^{\infty} a_n$$

converges and we write

$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} s_m = \lim_{m\to\infty} \sum_{n=1}^{m} a_n.$$

It is clear that convergence of a series is not affected by changes in a finite number of terms of the series (although the value of the sum is).

(1) **Cauchy's Criterion.** The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists N such that

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon \qquad \forall m, n, \quad m \geq n \geq N.$$

(Hint. Consider the sequence of partial sums $\{s_n\}$.)

(2) **The general term of a convergent series.** If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. The reciprocal does not hold.
(Hint. Use Cauchy's Criterion with $m = n$. For the counterexample, take $a_n = 1/n$.)

A series is said to be **absolutely convergent** when

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Note that in this case, since the partial sums are a non-decreasing sequence, this is equivalent to the existence of $M > 0$ such that

$$\sum_{n=1}^{m} |a_n| \leq M \qquad \forall m.$$

(3) Any absolutely convergent series is convergent. The reciprocal does not hold. (Hint. Use Cauchy's criterion. For the counterexample, use $a_n = (-1)^n/n$.)

(4) **The comparison test.** If

$$|a_n| \leq b_n \qquad \forall n$$

and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If, however,

$$0 \leq c_n \leq a_n \qquad \forall n,$$

and $\sum_{n=1}^{\infty} c_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
(Hint. For the first part, prove absolute convergence by showing that the partial sums are bounded.)

(5) **An example: the harmonic series.** Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

is convergent for $\alpha > 1$ and divergent (not convergent) for $0 \leq \alpha \leq 1$.
(Hint. For $\alpha > 1$ study the partial sums s_{2^n}. The case $\alpha = 2$ can be used as a template. For $\alpha \leq 1$, use the comparison test.)

(6) **The root test.** Let

$$\alpha = \limsup |a_n|^{1/n}.$$

Show that:

(a) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. (Hint. There exists $\beta < 1$ such that $|a_n| < \beta^n$ for large enough n. Why?)

(b) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. (Hint. Find a subsequence $\{a_{n_k}\}$ such that $|a_{n_k}|^{1/n_k} \to \alpha$. Argue that a_n does not converge to zero.)

(c) Study the examples $a_n = 1/n$ and $a_n = 1/n^2$ to show that the case $\alpha = 1$ is inconclusive.

(7) **The ratio test.**

(a) If

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then $\sum_{n=1}^{\infty} a_n$ converges. (Hint. There exist $\beta < 1$ and $N \in \mathbb{N}$ such that $|a_{n+1}| \leq \beta |a_n|$ for all $n \geq N$. Prove that $|a_{N+p} \leq \beta^p |a_N|$ for all $p \geq 1$.)

(b) If there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| \geq 1 \qquad \forall n \geq N,$$

then $\sum_{n=1}^{\infty} a_n$ diverges. (Hint. Show that a_n does not converge to zero.)

(c) Show that if

$$\left| \frac{a_{n+1}}{a_n} \right| \to 1,$$

nothing can be said about the convergence of the series.

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n z^n,$$

where $\{c_n\}$ is a given sequence of complex numbers and $z \in \mathbb{C}$ is taken as a variable. Given a power series, we consider the parameters

$$\alpha := \limsup |c_n|^{1/n} \in [0, +\infty] \qquad \text{and} \qquad R = \begin{cases} 1/\alpha & \text{if } \alpha \neq 0 \text{ and } \alpha \neq +\infty, \\ 0 & \text{if } \alpha = +\infty, \\ +\infty & \text{in } \alpha = 0. \end{cases}$$

Because of the following result, the value R is called the **radius of convergence** of the power series.

(8) The power series converges for all $z \in \mathbb{C}$ such that $|z| < R$ and diverges for all $z \in \mathbb{C}$ such that $|z| > R$. (Hint. Use the ratio test.)

(9) Study the convergence of the following power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sum_{n=0}^{\infty} z^n, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}.$$