MATH 612: CM4ES&FM

Spring'14 | Work'n'code day | February 28

AN EXPLORATION OF HOUSEHOLDER MATRICES

In this document norms are always 2-norms. Let $u \in \mathbb{C}^n$ be a unit vector $\|u\| = 1$. The Householder matrix associated to u is the matrix

$$H_u := I - 2u\,u^*.$$

Prove all of the following results ignoring what we have done for projectors. The proofs are really easy.

1. Show that H_u is hermitian and unitary and therefore $H_u^2 = I$.

2. Show that $H_u u = -u$ and that

$$H_u v = v \qquad \Longleftrightarrow \qquad v \perp u.$$

   (These results show that defines H_u is a symmetry operator. Can you see around what?) What is the characteristic polynomial of H_u? What is the determinant of H_u?

3. Write the formula to compute $H_u v$ using only vector operators (the matrix H_u is not built).

4. Let $\|y\| = 1$ and let x be not proportional to x. Consider the vector

$$u := \frac{1}{\big\|x + s\,\|x\|y\big\|}\big(x + s\,\|x\|y\big), \qquad s = +1 \text{ or } -1.$$

   Show that

$$H_u x = -s\,\|x\|y = \mp\|x\|y.$$

   (Hint. Prove that $\big\|x + s\,\|x\|y\big\|^2 = 2(x + s\,\|x\|y)^* x$.)

5. Show that if $u_1, \ldots, u_k$ are unit vectors, then

$$H = H_{u_k} \ldots H_{u_1}$$

   is a unitary matrix.

6. Write a program `y=householdertransf(U,x)` with the following specifications:

   - The input is an $n \times k$ matrix U whose k column vectors have unit norm, and a column vectors x;

- The output is

$$y = \underbrace{H_{u_k} \dots H_{u_2} H_{u_1}}_{H} x,$$

  where $u_1, \dots, u_k$ are the columns of U.

- The Householder matrices cannot be built. Only vector operations are allowed.
- The code should accept an $n \times m$ input matrix x and output Hx. (You can ignore this in a first version of the code.)

Find a way of testing that you built the right code.

7. With the function of the previous exercise (no modifications allowed), and being given a matrix U from which you theoretically build the product of Householder matrices

$$H = H_{u_k} \dots H_{u_1},$$

show a way to compute $H^{-1}x$.

Householder's method. Let A be a given $m \times n$ matrix and assume that A has maximum rank by columns. To better describe the algorithm, start by making a copy

$$A^{(1)} := A.$$

At the step k (there will be n steps, counted from 1 to n), we have a matrix $A^{(k)}$. Its k-th column is decomposed in the following form

$$\begin{bmatrix} b_k \\ c_k \end{bmatrix}, \qquad b_k \in \mathbb{R}^{k-1}, \qquad c_k = \begin{bmatrix} s_k \\ \times \\ \vdots \\ \times \end{bmatrix} \in \mathbb{R}^{m-k+1}.$$

(In particular, in the first step, c_1 is the entire first column of A.) We then compute

$$v_k := \frac{1}{\|c_k + \mathrm{sign}(s_k)\|c_k\|e_1\|}(c_k + \mathrm{sign}(s_k)\|c_k\|e_1), \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{m-k+1},$$

and consider the new matrix

$$A^{(k+1)} := H_{u_k} A^{(k)}, \qquad \text{where} \qquad u_k := \begin{bmatrix} 0 \\ v_k \end{bmatrix}.$$

The output of the method is the matrix $R = A^{(n)}$, which is uper triangular (why?) and a collection of vectors $u_1, \dots, u_n$, associated to a matrix

$$H = H_{u_n} \dots H_{u_1},$$

so that $HA = A^{(n)} = R$. Then $Q = H^*$ satisfies $A = QR$.