MATH 612
Computational methods for equation solving and function minimization – Week # 12

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Plan for this week

- Discuss any problems you couldn't solve from previous lectures
- We will cover part of Chapter 4 of the notes *Fundamentals of Optimization* by R.T. Rockafellar (University of Washington).
- We'll spend some time with realistic-looking problems.
- Coding assignment #4 is due next Monday

CONVEX SETS

Definition and the simplest examples

A set C is convex when

$$x_0, x_1 \in C \qquad \Longrightarrow \qquad (1-\tau)x_0 + \tau x_1 \in C \quad \forall \tau \in (0,1).$$

In other words, C contains all segments connecting points of C. Therefore, C contains the convex hull of any collection of points of C. (Recall that a set is closed when it contains all its limit points.)

Boxes. Let $I_1, \ldots, I_n$ be closed intervals, possibly unbounded. Then

$$I_1 \times I_2 \times \ldots I_n \subset \times \mathbb{R}^n \quad \text{is convex and closed.}$$

Also, if $b \in \mathbb{R}^n$, the half-space

$$\{x \in \mathbb{R}^n \,:\, x \cdot b \leq \alpha\} \quad \text{is convex and closed.}$$

Rules to build convex sets

- The finite intersection of convex sets is convex.
- If $C_1 \subset \mathbb{R}^n$ and $C_2 \subset \mathbb{R}^m$ are convex, then

$$C_1 \times C_2 \subset \mathbb{R}^{n+m} \quad \text{is convex.}$$

- If f is convex, then the level sets

$$\{x \in \mathbb{R}^n \,:\, f(x) \leq \alpha\}$$

  are convex (or empty).
- The affine image of a convex set is convex, that is, if $C \subset \mathbb{R}^n$ is convex and $A \in \mathbb{R}^{m \times n}$, then

$$\{Ax + b \,:\, x \in C\} \text{ is convex.}$$

More examples

- The positive orthant

$$\{x \in \mathbb{R}^n \; : \; x \geq 0\} = \{(x_1, \ldots, x_n) \; : \; x_i \geq 0 \quad \forall i\}$$

  is a closed convex set. (It is a particular case of a box.)
- The set

$$\{x \in \mathbb{R}^n \; : \; Ax = b\}$$

  is convex. Note how this set is the intersection of

$$\{x \; : \; Ax \leq b\} \cap \{x \; : \; -Ax \leq -b\}$$

- Euclidean balls

$$\{x \; : \; |x - x_0| \leq \alpha\}$$

  are convex.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the set

$$C = \{(x, u) \in \mathbb{R}^n \times \mathbb{R} \,:\, f(x) \le u\}$$

is convex.

Proof. If f is convex and $(x_0, u_0), (x_1, u_1) \in C$, then for all $\tau \in (0, 1)$

$$\begin{aligned} f((1-\tau)x_0 + \tau x_1) &\le (1-\tau)f(x_0) + \tau f(x_1) && (f \text{ is convex}) \\ &\le (1-\tau)u_0 + \tau u_1 && ((x_0, u_0), (x_1, u_1) \in C) \end{aligned}$$

and then $(1-\tau)(x_0, u_0) + \tau(x_1, u_1) \in C$. Reciprocally, let C be convex. Since $(x_0, f(x_0)), (x_1, f(x_1)) \in C$, then $(1-\tau)(x_0, f(x_0)) + \tau(x_1, f(x_1)) \in C$, that is,

$$f((1-\tau)x_0 + \tau x_1) \le (1-\tau)f(x_0) + \tau f(x_1).$$

LINEAR PROGRAMMING

Setting up the problem

In linear programming we minimize (or maximize) a linear function (which is therefore convex, and concave)

$$f(x) = x \cdot c$$

over a convex set delimited by linear inequalities or equalities

$$A_1 x \leq b_1, \qquad A_2 x_2 \geq b_2, \qquad A_3 x_3 = b_3.$$

When we write $x \leq y$, we mean

$$x_i \leq y_i \qquad \forall i.$$

All inequalities of the same kind

There are many forms of writing the LP problems. We will now explore some standard ways of writing them.

It is equivalent to have

$$A_1 x \leq b_1, \qquad A_2 x_2 \geq b_2, \qquad A_3 x_3 = b_3.$$

or

$$\begin{bmatrix} A_1 \\ -A_2 \\ A_3 \\ -A_3 \end{bmatrix} x \leq \begin{bmatrix} b_1 \\ -b_2 \\ b_3 \\ -b_3 \end{bmatrix}$$

LP problem in standard form

Traditionally a linear programming problem has been written in the following standard form:

$$\text{maximize} \qquad c \cdot x$$

subject to

$$Ax \leq b, \qquad x \geq 0.$$

This standard form has been motivated by the use of some particular methods that have been programmed for problems given in this particular format. Much software nowadays do the change for you. (See MATLAB's `linprog`.)

Change to standard form

- We have already seen how to move all inequalities and equalities to a joint set $Ax \leq b$.
- Moving from minimization to maximization is just a change of sign in the objective function.
- Sometimes the problem includes bounds for the unknowns

$$l \leq x \leq u.$$

  Then we use the variable $\widetilde{x} = x - l \geq 0$ and add inequalities $\widetilde{x} \leq u + l$.
- If we do not have lower bounds (for all or some variables), write

$$x = \widetilde{x} - \widehat{x}, \qquad \widetilde{x} \geq 0, \widehat{x} \geq 0$$

A transformation of the feasible set

Assume that our feasible set is given in standard form

$$Ax \leq b \qquad x \geq 0.$$

We can then add **slack variables** $s = b - Ax$ and write the feasible set in added variables

$$\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b, \qquad \begin{bmatrix} x \\ s \end{bmatrix} \geq 0.$$

This gives another standard way of writing the feasible set as a set of **non-negative solutions of a linear system**:

$$D\widehat{x} = b, \qquad \widehat{x} \geq 0.$$

With this form the number of variables has increased to the original number of variables plus the number of restrictions in standard form.

A toy example

Transform the feasible set

$$x_1 + x_2 + x_3 \leq 1, \qquad x_1 + x_2 \geq -1, \qquad x_1 \geq 0, \qquad x_3 \geq -1$$

to standard form and then to a set of non-negative solutions of a linear system.

Step 1. Let us first create non-negativity conditions for all variables:

$$x_1 = z_1, \qquad x_2 = z_2 - z_3, \qquad x_3 = z_4 + 1, \qquad z \geq 0.$$

We then rewrite the inequalities:

$$z_1 + z_2 - z_3 + z_4 \leq 0, \qquad z_1 + z_2 - z_3 \geq -1.$$

Step 2. We now write

$$z_1 + z_2 - z_3 + z_4 \leq 0, \qquad z_1 + z_2 - z_3 \geq -1$$

in the equivalent form

$$z_1 + z_2 - z_3 + z_4 \leq 0, \qquad -z_1 - z_2 + z_3 \leq 1.$$

This finishes the transformation of the feasible set to standard form

A toy example (3)

Introduction of slack variables.

$$z_5 = -z_1 - z_2 + z_3 - z_4 \geq 0, \qquad z_6 = 1 + z_1 + z_2 - z_3 \geq 0.$$

Therefore the feasible set can be rewritten as

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad z \geq 0$$

We finally need to recall how to obtain the original variables

$$x_1 = z_1, \qquad x_2 = z_2 - z_3, \qquad x_3 = z_4 + 1.$$

From a classic book...

Suppose General Motors makes a profit of \$200 on each Chevrolet, \$300 on each Buick, and \$500 on each Cadillac. These get 20, 17, and 14 miles per gallon, respectively, and Congress insists that the average car must get 18. The plant can assemble a Chevrolet in 1 minute, a Buick in 2 minutes, and a Cadillac in 3 minutes. What is the maximum profit in 8 hours (480 minutes)?

Another one...

Federal bonds pay 5%, municipals pay 6%, and junk bonds pay 9%. We can buy amounts x, y, z not exceeding a total of \$100,000. The problem is to maximize the interest, with two constraints:

- no more than \$20,000 can be invested in junk bonds, and
- the portfolio's average quality must be no lower than municipals, so $x \geq z$.