MATH 617. Introduction to Applied Mathematics II

Francisco–Javier Sayas

Spring 2012

Quiz #4 : interactive part

Q1. Conservation of energy

Show that if

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \text{in } \Omega \times [0, T]$$
$$\nabla u \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times [0, T]$$

and

$$E(t) := \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\int_\Omega |\nabla u|^2,$$

then $E(t)$ is constant in time.

Q2. Uniqueness from energy conservation

Show that the IBVP

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \Delta u + f && \text{in } \Omega \times [0, T] \\
\nabla u \cdot \mathbf{n} &= g && \text{on } \partial\Omega \times [0, T] \\
u(\,\cdot\,, 0) &= u_0 && \text{in } \Omega \\
\frac{\partial u}{\partial t}(\,\cdot\,, 0) &= v_0 && \text{in } \Omega
\end{aligned}$$

has *at most* one solution.

Hint. Subtract two possible solutions and use the energy argument of the previous exercise.

Q3. Eigenfunctions as IC

If

$$\begin{aligned} -\Delta\phi &= \omega^2\phi && \text{in } \Omega \\ \nabla\phi\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned} \qquad \begin{aligned} -\Delta\psi &= \rho^2\psi && \text{in } \Omega \\ \nabla\psi\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

write down the solution of

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta u && \text{in } \Omega\times[0,T] \\ \nabla u\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega\times[0,T] \\ u(\,\cdot\,,0) &= \phi && \text{in } \Omega \\ \frac{\partial u}{\partial t}(\,\cdot\,,0) &= \psi && \text{in } \Omega \end{aligned}$$

Hint. You do not need to argue with the entire eigenfunction expansion. Think that the expansion is limited to two terms. You do not need the eigenvalues to be different or the eigenfunctions to be normalized.

Quiz #3 : interactive part

Q1. Orthonormal sets...

1. Define what we mean by a complete orthonormal sequence $\{f_n\}$ in an inner product space.
2. State Parseval's equality.
3. If $f = 5f_3 - 2f_6$, what is $\|f\|$?

Q2. Gaussian solutions to the heat equation

Let $U(x,t;\xi) := \frac{1}{\sqrt{4\pi\kappa t}} \exp(-\frac{(x-\xi)^2}{4\kappa t})$ and

$$u(x,t) := \int_{-\infty}^{\infty} U(x,t;\xi) w(\xi) \mathrm{d}\xi.$$

Show that if $a \le w(\xi) \le b$ for all ξ, then

$$a \le u(x,t) \le b \qquad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

Q3. Energy arguments for diffusion equations

Consider the problem

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \phi(u) \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions $u(0,t) = u(1,t) = 0$, initial condition $u(x,0) = u_0(x)$ and $\kappa > 0$. We assume that the reaction term (which might be non-linear) satisfies

$$u\phi(u) \geq 0.$$

We define the total energy at time t as

$$E(t) = \frac{1}{2} \int_0^1 u(x,t)^2 \mathrm{d}x.$$

Show that $E(t) \leq E(0)$.

Quiz #2 : interactive part

Q1. Once again, give me the definitions of ...

1. $L^2(\Omega)$
2. its inner product
3. its associated norm

Q2. With an energy argument

Show that the BVP

$$-(py')' + q\,y = f \qquad \text{in } (a,b),$$

$$-p(a)y'(a) + y(a) = c_a, \qquad y'(b) = c_b$$

has at most one solution assuming that:

$$p(x) \geq p_0 > 0 \qquad q(x) \geq 0.$$

Hint. Everything is linear, so uniqueness is a question of having a look at the associated homogeneous problem. This means, take two solutions, subtract them, multiply the equation, integrate by parts, ...

Q3. Integration by parts and all that...

Let ϕ and ψ satisfy:

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega |\phi|^2 = 1,$$

$$-\Delta\psi = \mu\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega |\psi|^2 = 1,$$

with $\lambda \neq \mu$.

Compute...

$$\int_\Omega \nabla\phi \cdot \nabla\psi \qquad \text{and} \qquad \int_\Omega |\nabla\phi|^2$$

Hint. The energy arguments for looking at possible eigenvalues and orthogonality of eigenfunctions contain this extra information. You can use what you know about ϕ and ψ.

Q4. Back to orthonormal series

If $\{\phi_n\}$ is a complete *orthonormal* set in the space $\mathcal{C}(\overline{\Omega})$ w.r.t. the inner product

$$\int_\Omega f(\mathbf{x})\, g(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

and $f = 3\phi_2 - 5\phi_4$, compute

$$\int_\Omega f\, \phi_n, \quad n \geq 1, \qquad \int_\Omega |f|^2.$$

Quiz # 1 : interactive part

Q1. Write down

1. The definition of the space $L^2(a, b)$.
2. Its standard inner product.
3. The norm associated to such inner product.
4. What we mean when we write $\sum_{n=1}^{\infty} c_n f_n = f$ in $L^2(a, b)$.

Q2. Exercise

The sequence

$$f_n(x) := \sin\left(\frac{n\pi x}{L}\right), \qquad n \geq 1$$

is complete orthogonal in $L^2(0, L)$, where this space is endowed with the inner product

$$(f, g) = \int_0^L f(x)g(x)\mathrm{d}x.$$

Show that it is also complete orthogonal in the space of continuous functions on $[0, L]$, using the same inner product.

Hint. This is a complete no-brainer. You need to look at the definition.

Q3. Exercise

The sequence

$$f_n(x) := \sin\left(\frac{n\pi x}{L}\right), \qquad n \geq 1$$

is complete orthogonal in $L^2(0, L)$, where this space is endowed with the inner product

$$(f, g) = \int_0^L f(x)g(x)\mathrm{d}x.$$

Show that $\{f_n \,:\, n \geq 2\}$ is not complete in $L^2(0, L)$.

Q4. Exercise

The sequence

$$f_n(x) := \sin\left(\frac{n\pi x}{L}\right), \qquad n \geq 1$$

is complete orthogonal in $L^2(0, L)$, where this space is endowed with the inner product

$$(f, g) = \int_0^L f(x)g(x)\mathrm{d}x.$$

Give a good reason why infinitely many of the sine Fourier coefficients of $f \equiv 1$ have to be non-zero.

Hint. What happens to the corresponding series if only a finite number of coefficients are non-zero?

Q5. Look it up, it's in your notes...

We have a function whose Fourier sine coefficients

$$\widehat{f}(n) = \int_0^1 f(x)\, e^{-2\pi\imath nx} \mathrm{d}x$$

satisfy the inequality

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty.$$

What can you say about f and about the convergence of its Fourier series $\sum_n \widehat{f}(n)e^{2\pi\imath nx}$?