# MATH 672 – Vector spaces
# Presentation, August 28

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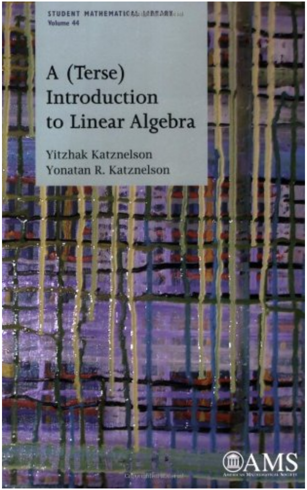
# The textbook




*A (terse) introduction to linear algebra*
by Yitzak Katznelson
& Yonatan R Katznelson

We will use the book for the lectures, following notation and taking exercises from the book

# A good help text

Google and download *Linear algebra done wrong* by Sergei Treil of Brown University

- Notation is NOT quite the same and we will be (slightly) more abstract in MATH672, but this book might help you review material at the matrix level.
- There's a link in the MATH672 website, in case you feel too tired to Google it yourself

# This will be a busy semester

**For a total of 1000 points ...**

1. **Two midterm exams (250 points each).** For each of them
   - An in-class part of the exam will be 150 points (60% of the grade of the exam)
   - A take-home part of the exam will be 100 points (40% of the grade of the exam)
2. **A two hour long in-class final exam (300 points)**
3. **A short 10/15 minute quiz every two weeks.**
   - Memorize definitions and simple proofs
   - Each exam will be worth 40 points
   - Only the highest 5 quizzes will count for your final grade

Homework will be assigned, but not collected. *You should be able to solve any exercise in the book.* There are not that many problems, but some of them are far from trivial.

# (Wise) words of advise

- **Learn the basic definitions and the main theorems by heart.** You cannot use what you don't know.
- Work on the problems right away. Don't let them pile up! If you can, make a small working group with some of your classmates.
- Be aware that **this is not an elementary linear algebra class**. You are supposed to be able to solve linear systems, to handle matrices,... (If you need help, ask, but don't expect the class to be about elementary stuff.)
- Read the assigned section of the book, before we get to class. *Lectures will be used as a help to read the book. There will not be selfcontained classnotes.*
- Don't be shy to use **office hours**. (Office hours TBA.)

# My expectations

- At the end of the semester, you need to know right away the main definitions and theorems of linear algebra
- In proofs, each equality/step should be justified (what exactly did we use?)
- Proofs and arguments have to be logically solid and well exposed. Mumbo-jumbo will not be accepted
- You should not have any problem with simple computations with rational arithmetic. (I will not waste any minute on this!)

# Attendance and deadline policy

Attendance to lectures is mandatory and will be checked on random dates. The students are expected to tell the instructor if they are going to miss a lecture, giving a good reason for it. Deadlines for the take-home part of the midterm exams will be strict.

# Academic honesty

As worded by the University of Delaware *All students must be honest and forthright in their academic studies. To falsify the results of one's research, to steal the words or ideas of another, to cheat on an assignment, or to allow or assist another to commit these acts corrupts the educational process. Students are expected to do their own work and neither give nor receive unauthorized assistance*. Any violation of this standard must be reported to the Office of Student Conduct. For more details, check http://www.udel.edu/stuguide/13-14/code.html Unless you are asked to do so in a concrete assignment, you cannot collaborate with your colleagues in assignments and projects. Cheating of any kind (even if the student does not take any advantage from it) will be grounds for an F grade.

# Today

1. We already had a look at the syllabus
2. **Assigned reading for Friday: read Section 1.1 of the book (groups and fields)**
3. **Assigned reading for Wednesday next week: Section 1.2 of the book (definition of vector field)**
4. We are getting linear systems out of our system....

Let's get rolling

# Can you see how these two things are the same?

$$\begin{bmatrix} 1 & -1 & 3 & 4 & 1 \\ 1 & -1 & 1 & 2 & 1 \\ 2 & -2 & 4 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 17 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 17 \end{bmatrix}$$

# Can you see how these two things are the same?

Find $(x_1, x_2, x_3, x_4, x_5)$ such that

$$\begin{bmatrix} 1 & -1 & 3 & 4 & 1 \\ 1 & -1 & 1 & 2 & 1 \\ 2 & -2 & 4 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 17 \end{bmatrix}$$

Find $x_1, x_2, x_3, x_4, x_5$ such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 17 \end{bmatrix}$$

# Gaussian elimination

Before...

$$\left[\begin{array}{ccccc|c} 1 & -1 & 3 & 4 & 1 & 11 \\ 1 & -1 & 1 & 2 & 1 & 5 \\ 2 & -2 & 4 & 6 & 1 & 17 \end{array}\right]$$

... and after ...

$$\left[\begin{array}{ccccc|c} \color{red}{\mathbf{1}} & -1 & 3 & 4 & 1 & 11 \\ 0 & 0 & \color{red}{\mathbf{-2}} & -2 & 0 & -6 \\ 0 & 0 & 0 & 0 & \color{red}{\mathbf{-1}} & 1 \end{array}\right]$$

1. All decisions are taken by the matrix. The RHS just suffers the consequences
2. Pivots are marked **in red**

# Consequences and terminology

$$\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 3 & 4 & 1 & 11 \\ 0 & 0 & \mathbf{-2} & -2 & 0 & -6 \\ 0 & 0 & 0 & 0 & \mathbf{-1} & 1 \end{array}\right]$$

- The system is solvable
- The variables $x_2$ **and** $x_4$ are free (they are parameters). In particular, the system is also solvable if we take them to vanish.
- The number of **pivots**, a.k.a., the rank, is three
- The number of **free variables**, a.k.a. the nullity, is two

# We are ready to solve...

Augmented matrix after Gaussian elimination

$$\left[\begin{array}{ccccc|c} \color{red}{1} & -1 & 3 & 4 & 1 & 11 \\ 0 & 0 & \color{red}{-2} & -2 & 0 & -6 \\ 0 & 0 & 0 & 0 & \color{red}{-1} & 1 \end{array}\right]$$

Equivalent system, with **free variables** sent to the RHS

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 \\ -6 \\ 1 \end{bmatrix} - \boldsymbol{x_2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \boldsymbol{x_4} \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 \\ -6 \\ 1 \end{bmatrix} + \boldsymbol{x_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{x_4} \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$$

**We'll resist the temptation though!**

# Gauss-Jordan elimination

**Goal**

Eliminate above the pivots and make them equal to one

Before...

$$\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 3 & 4 & 1 & 11 \\ 0 & 0 & \mathbf{-2} & -2 & 0 & -6 \\ 0 & 0 & 0 & 0 & \mathbf{-1} & 1 \end{array}\right]$$

... and after ...

$$\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 0 & 1 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & -1 \end{array}\right]$$

# How we have solved the problem

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

# Conclusions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \underbrace{x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\textbf{What is left}}$$

1. The vector in **red** is a particular solution (take $x_2 = x_4 = 0$)
2. **The part with the parameters** is the general solution of the homogeneous system. Why? GJ elimination with vanishing RHS gives exactly the same formula **except the particular solution is zero**

# Something else

This problem is always solvable:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This problem is always uniquely solvable:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- The structure of pivots and free variables does not depend on the RHS
- The free variables are not needed for existence of solution
- Without the free variables solution is unique
- The problems are always solvable because the pivots get all the way to the final row (rank=number of rows)

# Did you understand? Let's see

1. Repeat the process for the system

$$\begin{aligned} x + y + z + t &= 1 \\ x - y + z - t &= 2 \end{aligned}$$

2. Repeat the process for the system

$$x_1 + x_2 + x_3 + x_4 = -1$$

(Gauss–Jordan elimination needs exactly zero steps.)