MATH 672: Vector spaces

Fall'13 | Final exam Part 2/2 | December 9

1. (20 points) Define:
   (a) Subspace of a vector space
   (b) Linear operator between two vector spaces
   (c) Adjoint of a linear operator between two vector spaces
   (d) Adjoint of a linear operator in an inner product space

2. (20 points) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over $\mathbb{F}$ with $\dim\mathcal{V} = 4$ and $\dim\mathcal{W} = 6$. Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.
   (a) What is $\nu(T^*) - \nu(T)$?
   (b) Show that T is not surjective.
   (c) Show that T^* is not injective.
   (d) If T is injective, what is its rank?

3. (25 points) Let $\mathcal{H}$ be a complex inner product space and $T \in \mathcal{L}(\mathcal{H})$.
   (a) Show that T^*T is self-adjoint.
   (b) Show that $\ker(T^*T) = \ker T$. (Hint. $\langle T^*Tu, u\rangle = \langle Tu, Tu\rangle$.)
   (c) Show that the operators $\frac{1}{2}(T + T^*)$ and $\frac{1}{2\imath}(T - T^*)$ are self-adjoint.

4. (20 points) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be subspaces of a seven dimensional vector space $\mathcal{V}$ and assume that $\dim\mathcal{V}_1 = 3$ and $\dim\mathcal{V}_2 = 4$. Tabulate all possibilities of dimensions of $\mathcal{V}_1 \cap \mathcal{V}_2$ and relate them to all possibilities of dimensions of $\mathcal{V}_1 + \mathcal{V}_2$. Is the sum direct in any of the cases?

5. (30 points) Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be surjective and

$$\dim\mathcal{V} = n > m = \dim(\mathcal{W}).$$

Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ be a basis of $\mathcal{V}$, where $\ker T = \text{span}[v_{m+1}, \ldots, v_n]$. Finally, let $\mathbf{w} = \{w_1, \ldots, w_m\}$ be given by

$$w_j = Tv_j, \qquad j = 1, \ldots, m.$$

   (a) Show that $\mathbf{w}$ is a basis for $\mathcal{W}$.
   (b) Consider now the operator $U \in \mathcal{L}(\mathcal{W}, \mathcal{V})$ given by

$$Uw_j = v_j, \qquad j = 1, \ldots, m.$$

   Show that $TUw = w$ for all $w \in \mathcal{W}$.
   (c) Show that U is injective but not surjective.

(d) Compute UTv_j for all j. (Hint. There are two groups of v_j vectors.)

(e) Write down the matrix for T in the given bases.

(f) Write down the matrix for U in the given bases.

6. (15 points) Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space $\mathcal{V}$ and let

$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \qquad j = 1, \ldots, n,$$

for some given coefficients $a_{ij} \in \mathbb{F}$. Show that $\{w_1, \ldots, w_n\}$ is a basis for $\mathcal{V}$ if and only if the matrix A is invertible.

7. (30 points) Let $\mathcal{H}$ be an inner product space, $\{q_1, \ldots, q_k, p_1, \ldots, p_m\}$ be an orthonormal basis and let

$$P_1 u = \sum_{j=1}^{k} \langle u, q_j \rangle q_j, \qquad P_2 u = \sum_{j=1}^{m} \langle u, p_j \rangle p_j.$$

(a) Show that the operators P_1 and P_2 are self-adjoint.

(b) Show that $P_1 P_2 = 0$ and $P_2 P_1 = 0$.

(c) Show that if $\lambda_1, \lambda_2 \in \mathbb{C}$, then the operator

$$T = \lambda_1 P_1 + \lambda_2 P_2.$$

is normal.

(d) Give necessary and sufficient conditions for T to be self-adjoint.

(e) Compute minP_T.

(f) Compute $\det T$.

8. (20 points) Let $T \in \mathcal{L}(\mathcal{V})$ be such that its matrix representation with respect to the basis $\{v_1, \ldots, v_6\}$ is

$$A = \begin{bmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & 1 & 3 & \\ & & & & 1 & 3 \end{bmatrix}$$

(a) Show that we cannot define an inner product in $\mathcal{V}$ such that T is self-adjoint.

(b) Find a basis for $\mathcal{V}$ such that the associated matrix is

$$\begin{bmatrix} 3 & 1 & & & & \\ & 3 & 1 & & & \\ & & 3 & & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \\ & & & & & 2 \end{bmatrix}$$

(c) Assume that $P \in \mathbb{R}_7[x]$ satisfies $P(A) = 0$ and $P(4I) = 0$. What is P?

(d) Show that $\{v_3, Tv_3, T^2v_3, v_4, Tv_4, T^2v_4\}$ is a basis for $\mathcal{V}$ and write down the matrix representing T with respect to this basis.

MATH 672: Vector spaces

Fall'13 — Final exam Part 2/2 (Plan B) — December 9

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1. (10 points) Show that $\{1+x+x^2+x^3, x+x^2+x^3, x^2+x^3, x^3\}$ is a basis for $\mathbb{R}_3[x]$.

2. (15 points) Consider the sets

$$\begin{aligned} \mathcal{V}_1 &= \{A \in \mathcal{M}(2;\mathbb{R}) : e^{Tr}Ae = 0\}, \qquad \text{where } e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \mathcal{V}_2 &= \{A \in \mathcal{M}(2;\mathbb{R}) : \operatorname{trace} A = 0\}. \end{aligned}$$

   Find a basis for $\mathcal{V}_1 \cap \mathcal{V}_2$.

3. (10 points) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be subspaces of $\mathbb{R}_6[x]$, both of them of dimension three. What are the possibilities for the dimensions of $\mathcal{V}_1 \cap \mathcal{V}_2$ and $\mathcal{V}_1 + \mathcal{V}_2$?

4. (10 points) Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be such that $\rho(T) = 4$, $\nu(T) = 2$ and T is surjective. What are the dimensions of $\mathcal{V}$ and $\mathcal{W}$?

5. (15 points) The characteristic polynomial of a matrix A is $\chi_A(\lambda) = (2-\lambda)^2(3-\lambda)(4+\lambda)$ and we know that A is diagonalizable. What is minP_A? Find the coefficients that make

$$A^3 = aA^2 + bA + cI.$$

6. (20 points) In the space $\mathbb{R}^3$ we consider the following inner product

$$\langle [x_1, x_2, x_3], [y_1, y_2, y_3] \rangle = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3) + x_2y_2 + x_3y_3$$

   and the subspace $\mathcal{V} = \{[x_1, x_2, x_3] : x_1 + x_3 = 0\}$.

   (a) Compute $\|[1,2,3]\|$.

   (b) Compute an orthonormal basis for $\mathcal{V}$.

   (c) Compute the orthogonal projection of $[1,1,1]$ onto $\mathcal{V}$.

   (d) Compute a basis for $\mathcal{V}^\perp$.

7. (40 points) Let $\{v_1, v_2, v_3, v_4\}$ be a basis for a real vector space $\mathcal{V}$, and $T \in \mathcal{L}(\mathcal{V})$ be given by

$$Tv_1 = 2v_1 + v_2, \qquad Tv_2 = 2v_2, \qquad Tv_3 = 2v_3, \qquad Tv_4 = v_3 + 2v_4.$$

   (a) Find the matrix representation of T in this basis.

   (b) Compute $T(v_1 - 2v_2 + 3v_3 - 4v_4)$ using the matrix in (a).

   (c) Show that $T \in \mathbf{GL}(\mathcal{V})$.

(d) Compute all eigenvalues and eigenvectors of A. (Warning. I am asking for all the eigenvectors, not for a basis.)

(e) Compute all eigenvalues and eigenvectors of T.

(f) Is T diagonalizable?

(g) Compute minP_T.

(h) Compute minP_{T,v_1} and minP_{T,v_2}.

8. (30 points) Consider the following three bases of $\mathbb{R}_2[x]$:

$$\mathbf{p} = \{1, 1+x, 1+x^2\}, \qquad \mathbf{e} = \{1, x, x^2\}, \qquad \mathbf{q} = \{1+x+x^2, x+x^2, x^2\}$$

Let $C_{\mathbf{p}}, C_{\mathbf{e}}$, and $C_{\mathbf{q}}$ be the associated coordinate maps and let $p(x) = 1 + x + x^2$.

(a) Compute $C_{\mathbf{p}}p$, $C_{\mathbf{e}}p$, and $C_{\mathbf{q}}p$.

(b) Compute the matrices for the following change of bases: $\mathbf{p}$ to $\mathbf{e}$, $\mathbf{e}$ to $\mathbf{q}$, $\mathbf{p}$ to $\mathbf{q}$.

(c) Verify the three matrices of (b) using the results of (a).

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Fall'13 — Final exam Part 1/2 (Plan B) — Due December 9

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1. (15 points) In the space $\mathbb{R}_2[x]$ we consider the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)x^2 dx$$

and the subspace

$$\mathcal{V} = \{p \in \mathbb{R}_2[x] \, : \, \int_{-1}^{1} x^3 p(x) dx = 0\}.$$

(a) Find a basis for $\mathcal{V}$.

(b) Use it to find an orthonormal basis for $\mathcal{V}$.

(c) If $\mathcal{W} = \text{span}[x]$, show that $\mathcal{W}^{\perp} = \mathcal{V}$.

2. (35 points) Let $\mathcal{V}$ be a 3-dimensional space, $\{v_1, v_2, v_3\}$ be a basis for $\mathcal{V}$ and $T \in \mathcal{L}(\mathcal{V})$ be given by

$$Tv_1 = 2v_1 + v_2 - v_3, \qquad Tv_2 = 3v_2 + v_3, \qquad Tv_3 = 4v_3.$$

(a) Find the matrix representation of T in the given basis. (Let us call it A.)

(b) Compute $T(v_1 - v_2 + \frac{1}{2}v_3)$ in two different ways: using the relations given at the beginning, and using the matrix representation.

(c) Compute $\det T$ and χ_T.

(d) Compute all eigenvalues and eigenvectors for A. Use them to compute all eigenvalues and eigenvectors of T.

(e) Is T diagonalizable?

(g) Compute minP_T.

(h) Compute minP_{T,v_3} and minP_{T,v_2}.

3. (15 points) In $\mathbb{R}^3$ we consider the dot product

$$\langle [x_1, x_2, x_3], [y_1, y_2, y_3] \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

the vectors $u_1 = [1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]$ and $u_2 = [1/\sqrt{2}, 1/\sqrt{2}, 0]$, and $\mathcal{W} = \text{span}[u_1, u_2]$.

(a) Show that $\{u_1, u_2\}$ is an orthonormal basis for $\mathcal{W}$.

(b) Find the orthogonal projection of $[1, 1, 1]$ onto $\mathcal{W}$.

(c) Find the orthogonal projection of $[1, 1, 0]$ onto $\mathcal{W}$.

4. (15 points) Show that the sum of the subspaces

$$\mathcal{V}_1 = \{[x, y, z] \,:\, x + y + z = 0\}, \qquad \mathcal{V}_2 = \{[x, y, z] \,:\, x = y = z\}$$

is direct. Find bases for $\mathcal{V}_1$ and $\mathcal{V}_2$.

5. (20 points) In $\mathbb{R}_3[x]$, we consider the sets

$$\begin{aligned} \mathbf{p} &= \{1 + x + x^2 + x^3, 1 + x^2 - x^3, x^2 + 2x^3, 3x^2\}, \\ \mathbf{q} &= \{x^3, x^2, x, 1\}, \end{aligned}$$

and the polynomial $r(x) = 2 + x^3$.

(a) Show that $\mathbf{p}$ is a basis for $\mathbb{R}_3[x]$.

(b) If $C_{\mathbf{p}} : \mathbb{R}_3[x] \to \mathbb{R}^4_c$ is the associated coordinate map, find $C_{\mathbf{p}} r$.

(c) Find $C_{\mathbf{q}} r$.

(d) Compute the matrix for the change of variables from $\mathbf{p}$ to $\mathbf{q}$. Verify your result using the computation of (b) and (c).

6. (20 points) Consider the map $T : \mathbb{R}_2[x] \to \mathcal{M}(2, \mathbb{R})$ given by

$$Tp(x) = \begin{bmatrix} p(0) & p(1) \\ p(1) & p'(0) \end{bmatrix}.$$

(a) Compute the matrix A that represents T in canonical bases for both spaces.

(b) Compute the matrix B that represents T in the following bases:

$$\mathbf{p} = \{1, 1 + x, 1 + x + x^2\} \qquad \text{for } \mathbb{R}_2[x],$$

$$\mathbf{m} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \qquad \text{for } \mathcal{M}(2; \mathbb{R}).$$

(c) Compute the matrix P for the change of basis from canonical to $\mathbf{p}$ in $\mathbb{R}_2[x]$.

(d) Compute the matrix Q for the change of basis from $\mathbf{m}$ to canonical in $\mathcal{M}(2; \mathbb{R})$.

(e) If you had your computations right in the previous four items, you should be able to verify that

$$A = QBP.$$

Do it.

7. Consider the matrices

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) Compute $A = PDP^{-1}$.

(b) Compute χ_A.

(c) Compute $\det A$.

(d) Compute all eigenvalues and eigenvectors for A. (As usual, eigenvectors should be given as spans.)

(e) Identify the subspaces for eigenvectors of A with spans of groups of columns of P.

(f) Justify the previous find with a theoretical argument.

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Fall'13 — Final exam Part 1/2 — Due December 9

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1. (15 points) Let $\mathcal{V}$ be a finite dimensional space and $T \in \mathcal{L}(\mathcal{V})$. Show that

$$\chi_T(0) = 0 \qquad \Longleftrightarrow \qquad \mathrm{minP}_T(0) = 0 \qquad \Longleftrightarrow \qquad \det T = 0.$$

2. (20 points) Consider the $n \times n$ matrix

$$A = \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}$$

(a) Compute minP_A.

(b) Find a vector $v \in \mathbb{F}_c^n$ such that $\mathrm{minP}_{A,v} = \mathrm{minP}_A$.

(c) Assuming that $\mu \neq 0$, find an explicit formula for A^{-1} and justify it. (Hint. By checking the lowest dimensional cases, it is easy to guess a general formula.)

(d) Compute $\mathrm{minP}_{A^{-1}}$.

3. (10 points) In $\mathbb{C}_c^n$, we consider the Euclidean (dot) inner product,

$$\langle u, v \rangle = \sum_j \overline{v}_j u_j.$$

Let now $q_1, \dots, q_k \in \mathbb{C}_c^n$ and let $Q \in \mathcal{M}(n \times k; \mathbb{C})$ be the matrix whose columns are the vectors $\{q_j\}$.

(a) Show that $\{q_1, \dots, q_k\}$ is an orthonormal set if and only if $Q^*Q = I_k$.

(b) If $k = n$, show that $\{q_1, \dots, q_n\}$ is an orthonormal set if and only if $Q^* = Q^{-1}$.

4. (10 points) In $\mathbb{R}_3[x]$, we consider the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Let then

$$\mathcal{H}_1 = \{p \in \mathbb{R}_3[x] \,:\, p(1) = 0\}, \qquad \mathcal{H}_2 = \{p \in \mathbb{R}_3[x] \,:\, p'(1) = 0\}.$$

(a) Find an orthonormal basis for $\mathcal{H}_1 \cap \mathcal{H}_2$. (Show how to compute everything, but do not show all details in computation of integrals.)

(b) Find the orthogonal projection of $p(x) = x^2$ onto $\mathcal{H}^1 \cap \mathcal{H}^2$.

5. (15 points) Let $\mathcal{H}$ be a finite dimensional inner product space.

(a) Give a direct proof of the following result:

$$\text{range}(T) = (\ker T^*)^\perp.$$

(b) Show that if $\mathcal{W}$ is a subspace of $\mathcal{H}$, then

$$\dim\mathcal{W} + \dim\mathcal{W}^\perp = \dim\mathcal{H}.$$

(c) Show that

$$\rho(T) = \rho(T^*)$$

using the previous results.

6. (15 points) In $\mathbb{R}_3[x]$, we consider the following functionals

$$\langle p, \psi_j \rangle = \frac{1}{j!} p^{(j)}(1), \qquad j = 0, 1, 2, 3,$$

the canonical basis $\mathbf{e} = \{1, x, x^2, x^3\}$ and its dual basis $\mathbf{e}^* = \{e_0^*, e_1^*, e_2^*, e_3^*\}$.

(a) Show that $\mathbf{q}^* = \{\psi_0, \psi_1, \psi_2, \psi_3\}$ is a basis for $\mathbb{R}_3[x]^*$.

(b) Write the matrix for the change of basis from $\mathbf{e}^*$ to $\mathbf{q}^*$.

(c) Using the result of (b), find the coefficients of the decomposition

$$e_2^* = \sum_{j=0}^{3} c_j \psi_j.$$

7. (15 points) In $\mathcal{M}(n;\mathbb{R})$ we consider the inner product

$$\langle A, B \rangle = \sum_{i,j} a_{ij} b_{ij} = \text{trace}(B^{Tr} A)$$

and the operator $TA = A^{Tr}$.

(a) Show that T is self-adjoint.

(b) Give a direct argument (no characteristic polynomial involved), to show that the only possible eigenvalues of T are ± 1.

(c) Use these facts and the spectral theorem to prove that $\mathcal{M}(n;\mathbb{R}) = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$, where $\mathcal{H}_1$ is the set of symmetric matrices and $\mathcal{H}_{-1}$ is the set of skew-symmetric matrices, and that the sum is orthogonal.

8. (20 points) Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator on a complex inner product space $\mathcal{H}$, and let $P \in \mathbb{R}[x]$. Show that $P(T)$ is also selfadjoint. If T is normal, is also $P(T)$ normal?

MATH 672: Vector spaces

Fall'13 | Quiz # 8 | Due December 2

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1. (30 points) Let $T \in \mathcal{L}(\mathcal{V})$, $\{v_1, \ldots, v_n\}$ be a basis for $\mathcal{V}$ and assume that

$$Tv_j = v_1 + v_2 + \ldots + v_n \qquad \forall j.$$

   (a) Show that $v_1 + \ldots + v_n$ is an eigenvector.
   (b) Compute $\nu(T)$.
   (c) Using the results of (a) and (b) (no computations are required), give χ_T.
   (d) Show that T is diagonalizable.
   (e) Compute minP_{T,v_j} for all j. (Hint. This is very easy using the definition.)
   (f) Write (and argument) what minP_T is.

2. (15 points) Let A be such that

$$A^2 = 4A - 2I.$$

   (a) What is the characteristic polynomial of A? (Prove your assertion.)
   (b) What are the possible minimal polynomials of A?
   (c) Show that either A is diagonal or it is not diagonalizable.

3. (10 points) Let $T \in \mathcal{L}(\mathcal{V})$ be given by its action on a basis

$$Tv_1 = 2v_1, \qquad Tv_2 = \alpha v_1 + 2v_2, \qquad Tv_3 = \beta v_1 + \gamma v_2 + 5v_3.$$

   Show that T is diagonalizable if and only if $\alpha = 0$. (Hint. There is no need to compute eigenvectors. You just need to count how many linearly independent eigenvectors there are.)

4. (10 points) In $\mathbb{R}_3[x]$ we consider the inner product

$$\langle p, q \rangle = \int_0^1 x p(x) q(x) dx.$$

   Compute a basis for $\{1, x\}^\perp$.

5. (15 points) Let $\mathcal{H}$ be a complex inner product space endowed with the product $\langle \cdot, \cdot, \rangle$. Let $\{v_1, \ldots, v_k\}$ be linearly independent vectors and consider the $k \times k$ matrix A with elements

$$a_{ij} = \langle v_i, v_j \rangle.$$

   (This is called a Gram matrix.)

(a) Show that A is Hermitian. (This is defined as $\overline{A}^{Tr} = A$.)

(b) If $c \in \mathbb{C}^n_c$, find $v \in \mathcal{H}$ such that

$$\|v\|^2 = \overline{c}^{Tr} Ac.$$

Show that $\overline{c}^{Tr} Ac \geq 0$ and $\overline{c}^{Tr} Ac = 0$ if and only if $c = 0$.

(c) By looking at the matrix A, how can you know if the set $\{v_1, \dots, v_k\}$ is orthogonal/orthonormal?

6. (10 points) **The parallelogram law.** Show that if $\mathcal{H}$ is an inner product space with inner product $\langle \cdot, \cdot \rangle$, then its associated norm $\| \cdot \|$ satisfies:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

7. (10 points) **The real polarization formula.** Let $\mathcal{H}$ be a real vector space endowed with a norm $\|\cdot\}$ satisfying the parallelogram law (in addition to the axioms that define a norm):

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Show that

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)$$

is an inner product in $\mathcal{H}$ whose associated norm is $\| \cdot \|$. (You have to prove that the bracket $\langle \cdot, \cdot \rangle$ satisfies all the axioms that define a real inner product.)

MATH 672: Vector spaces

Fall'13 | Quiz # 8 (Plan B) | Due December 2

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1. (10 points) Consider the following inner product in $\mathbb{R}_2[x]$

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

and the set

$$B = \{1, x - \tfrac{1}{2}, (x - \tfrac{1}{2})^2 - \tfrac{1}{12}\}$$

(a) Show that B is orthogonal.

(b) Use it to find an orthonormal basis of $\mathbb{R}_2[x]$.

2. (15 points) In $\mathbb{R}^3$ with the dot product

$$\langle [x_1, x_2, x_3], [y_1, y_2, y_3] \rangle = \sum_j y_j x_j$$

give a basis of the subspaces

$$\mathcal{V}_1 = \{z \in \mathbb{R}^3 \, : \, z \perp [1, 2, 3]\},$$
$$\mathcal{V}_2 = \{z \in \mathbb{R}^3 \, : \, z \perp [1, 1, 1]\},$$
$$\mathcal{V}_3 = \mathcal{V}_1 \cap \mathcal{V}_2.$$

3. (20 points) Let T be an operator in $\mathcal{V}$ (a general three dimensional vector space, not $\mathbb{F}^3$) such that

$$Tv_1 = 2v_1, \qquad Tv_2 = 2v_2 + v_1, \qquad Tv_3 = -v_2 + 4v_3,$$

where $\{v_1, v_2, v_3\}$ is a basis for $\mathcal{V}$.

(a) Write down the matrix A that is associated to T with respect to this basis.

(b) Find all the eigenvalues and eigenvectors of A.

(c) Use the answer to the previous question to give all eigenvectors of T.

(d) Is T diagonalizable?

4. (15 points) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces with $\dim \mathcal{V} = 4$ and $\dim \mathcal{W} = 3$. Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Tabulate all possible values of the rank and the nullity of T. Are there any cases where T is injective/surjective?

5. (15 points) Find a basis for the intersection

$$\{A \in \mathcal{M}(2; \mathbb{R}) \,:\, A^{Tr} = A\} \cap \{A \in \mathcal{M}(2; \mathbb{R}) \,:\, \mathrm{trace}(A) = 0\}.$$

6. (15 points) Find a basis for

$$\{p \in \mathbb{R}_3[x] \,:\, p(1) = p(-1)\}.$$

7. (15 points) Find the matrix for the change of basis from $\mathbf{p} = \{1, 1+x, 1+x+x^2\}$ to $\mathbf{q} = \{1 - x + x^2, 1, -x\}$ in $\mathbb{R}^2[x]$. Check your result with $p(x) = 1 + x^2$

$$1 - (1+x) + (1+x+x^2) = p(x) = (1 - x + x^2) + (-1)(-x).$$

MATH 672: Vector spaces

Fall'13 | Second midterm exam Part 2/2 (Plan B) | November 15

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1. (25 points) Let $T \in \mathcal{L}(\mathcal{V})$ be given by

$$Tv_1 = 2v_1, \qquad Tv_2 = 2v_2 + v_1, \qquad Tv_3 = 2v_3,$$

where $\{v_1, v_2, v_3\}$ is a basis for $\mathcal{V}$. Compute:

(a) The characteristic polynomial of T and the determinant of T.

(b) All eigenvectors of T. (Note. The eigenvectors of T are related to the eigenvectors of its associated matrix, but they are not the same, since they are elements of $\mathcal{V}$, not of $\mathbb{F}^3$.)

(c) The minimal polynomial of T.

(d) The minimal polynomial of v_1 and v_2.

2. (25 points) Consider the following two bases for $\mathbb{R}_1[x]$:

$$\mathbf{p} = \{1 + x, 1 - x\}, \qquad \mathbf{q} = \{3 + 2x, 1 + 2x\}.$$

   (a) Build the matrix the transforms polynomials expressed in the basis $\mathbf{q}$ to polynomials expressed in the basis $\mathbf{p}$. (Hint. It is easier if you use an intermediate basis.)

   (b) Check your result using the following example

$$2(3 + 2x) - 1(1 + 2x) = \tfrac{7}{2}(1 + x) + \tfrac{3}{2}(1 - x).$$

3. (25 points) Let $T : \mathbb{R}_2[x] \to \mathcal{M}(2;\mathbb{R})$ be given by

$$Tp(x) = \begin{bmatrix} p(0) & p(1) \\ p'(0) & p'(1) \end{bmatrix}$$

   Write its coordinate matrix in canonical basis for both spaces. For $\mathcal{M}(2;\mathbb{R})$, the canonical basis is defined by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

   Show that T is injective.

4. (25 points) Let $T : \mathcal{M}(2;\mathbb{R}) \to \mathcal{M}(2;\mathbb{R})$ be given by

$$TA = \tfrac{1}{2}A + A^{Tr}.$$

   Compute the associated matrix in the following basis for $\mathcal{M}(2 : \mathbb{R})$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

   Is T diagonalizable?

5. (20 points) Let

$$\mathcal{V}_1 = \{p \in \mathbb{R}_2[x] \, : \, p(1) = 0\}, \qquad \mathcal{V}_2 = \{p \in \mathbb{R}_2[x] \, : \, p(0) = p'(0)\}.$$

   Find a basis for $\mathcal{V}_1 \cap \mathcal{V}_2$. Knowing this, and without any additional computation, give a basis for $\mathcal{V}_1 + \mathcal{V}_2$. (Hint. Look at dimensions.)

6. (15 points) Let $\dim \mathcal{V} = 5$ and $\mathcal{V}_1, \mathcal{V}_2$ be subspaces of $\mathcal{V}$ with $\dim \mathcal{V}_1 = 4$ and $\dim \mathcal{V}_2 = 3$. What are all the possible values for the dimension of $\mathcal{V}_1 \cap \mathcal{V}_2$? What are the corresponding dimensions for $\mathcal{V}_1 + \mathcal{V}_2$? (Warning. The values are related and it is relevant to note that $\dim \mathcal{V} = 5$.)

7. (15 points) Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ where $\dim \mathcal{V} = 4$ and $\dim \mathcal{W} = 3$. Tabulate all possible values for the rank and nullity of T. Mark what cases correspond to injective and/or surjective operators.

MATH 672: Vector spaces

Fall'13 — Second midterm exam Part 2/2 — November 15

1. (20 points) Define:
   (a) Linear functional on a vector space $\mathcal{V}$.
   (b) The dual space of a vector space $\mathcal{V}$.
   (c) Alternating $k-$linear form in a vector space $\mathcal{V}$.
   (d) The transpose of an operator $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

2. (15 points) State (giving all hypotheses):
   (a) Cayley-Hamilton's Theorem.
   (b) The relation between the nullity and the rank of an operator $\mathcal{L}(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional.
   (c) The result that relates the dimensions of the intersection and the sum of two subspaces of a finite dimensional space.

3. (25 points) Let T be given by the following relations

$$Tv_1 = 2v_1, \qquad Tv_2 = 2v_2 + v_1, \qquad Tv_3 = -v_3, \qquad Tv_4 = -v_4 + v_3, \qquad Tv_5 = 2v_5,$$

   where $\{v_1, \ldots, v_5\}$ is a basis for $\mathcal{V}$.

   (a) Find a basis for $\mathcal{V}$ such that the associated matrix is

$$A = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & 1 & 2 & & \\ & & & -1 & \\ & & & 1 & -1 \end{bmatrix}$$

   (The numbers that are not displayed are zero.)

   (b) Find all eigenvectors of A. Relate them to eigenvectors of T. (Recall that eigenvectors are organized in subspaces, so eigenvectors should be returned as spans.)
   (c) Find the minimal polynomial for v_2 and v_5.
   (d) Give a reasoned guess of what the minimal polynomial of T is.
   (e) Is T diagonalizable? Why/why not?

4. (20 points) Find the subspace $\mathcal{V} \subset \mathbb{R}_3[x]$ such that

$$\mathcal{V}^\perp = \text{span}[\phi_0, \phi_1], \qquad (p, \phi_0) = p(0), \qquad (p, \phi_1) = p(1).$$

5. (20 points) Consider the operator $T : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = (1+x)p(x)$. Let $\mathbf{e} = \{1, x\}$ and $\mathbf{f} = \{1, x, x^2\}$ be the respective canonical bases for $\mathbb{R}_1[x]$ and $\mathbb{R}_2[x]$, and let $\mathbf{e}^*$ and $\mathbf{f}^*$ be the corresponding dual bases for $\mathbb{R}_1[x]^*$ and $\mathbb{R}_2[x]^*$. What is the matrix for T^* written in the bases $\mathbf{f}^*$ and $\mathbf{e}^*$?

6. (20 points) If $T = R^{-1}SR$, where $R \in \mathbf{GL}(\mathcal{V})$, prove that $P(T) = R^{-1}P(S)R$ for every polynomial $P \in \mathbb{F}[x]$. Conclude that the minimal polynomials for T and S are the same.

7. (20 points) Let $S \in \mathcal{L}(\mathcal{V})$. Then

$$\ker S^k \subseteq \ker S^{k+1} \qquad \forall k.$$

   Why?

   Assume now that

$$\ker S^k = \ker S^{k+1},$$

   pick $u \in \mathcal{V}$ such that $S^{k+2}u = 0$, and define $v = Su$. Then $v \in \ker S^{k+1}$. Why?

   This implies that $S^{k+1}u = 0$. Why?

   We have thus proved that if $\ker S^k = \ker S^{k+1}$, then $\ker S^{k+1} = \ker S^{k+2}$. Explain how.

8. (20 points) Let A be a square matrix such that $A \neq 0$ and $A \neq I$. Assume that

$$A^2 - A = 0.$$

   Show that

$$\chi_A(x) = (-1)^n x^k (x-1)^{n-k}, \qquad 1 \leq k \leq n-1.$$

MATH 672: Vector spaces

Fall'13 Second midterm exam Part 1/2 (Plan B) Due November 18

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1. (15 points) Find a basis for

$$\{p \in \mathbb{R}_3[x] \,:\, p(0) = p'(1) = 0\}.$$

2. (15 points) Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

   and consider the following linear functional in the space $\mathcal{M}(2, \mathbb{R})$:

$$(A, \psi^*) = \text{trace}(A) = a_{11} + a_{22}.$$

   Show that $\psi^* \in S^\perp$. Compute the dimension of span$[S]$. Finally, use a theoretical argument to show that $\{\psi^*\}$ is a basis for $S^\perp$.

3. (20 points) Consider the linear operator $T \in \mathcal{L}(\mathbb{R}_2[x])$ given by

$$Tp(x) = xp'(x) + p(x).$$

(a) Write down its matrix in canonical basis $\mathbf{e} = \{1, x, x^2\}$. (Let's call it A.)

(b) Write down its matrix in basis $\mathbf{q} = \{1, 1+x, 1+x+x^2\}$. (Let's call it B.)

(c) Write down the matrix for the change of basis from $\mathbf{q}$ to $\mathbf{e}$. (Let's call it P.)

(d) One of the following identities holds:

$$A = PBP^{-1} \qquad \text{or} \qquad A = P^{-1}BP.$$

Which one? Give a theoretical argument without computing P^{-1}. (Note that you can check the result with a computation to see if you were right.)

4. (15 points) Let $\Psi : \mathbb{R}_2[x] \times \mathbb{R}_2[x] \to \mathbb{R}$ be given by

$$\Psi(p(x), q(x)) = \int_{-1}^{1} p(x)q'(x)dx.$$

Show that it is bilinear. Give an example that shows that Ψ is neither symmetric nor alternating.

5. (20 points) Let $T \in \mathcal{L}(\mathcal{V})$ be given by

$$Tv_1 = v_2, \quad Tv_2 = v_3, \quad Tv_3 = v_4, \quad Tv_4 = -v_1 - 4v_2 - 6v_3 - 4v_4.$$

where $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathcal{V}$.

(a) Write down its matrix with respect to this basis.

(b) Compute $\det T$.

(c) Show that $T \in \mathbf{GL}(\mathcal{V})$.

(d) Compute the characteristic polynomial of T.

(e) Compute all possible eigenvalues and eigenvectors of T.

6. (15 points) Let $T : \mathcal{M}(2 \times 3, \mathbb{R}) \to \mathbb{R}_6[x]$ be such that $\ker(T)$ is three dimensional. What is the rank of T? Is T onto? Is T injective?

MATH 672: Vector spaces

Fall'13 — Second midterm exam Part 1/2 — Due November 18

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1. (15 points) Let $\{v_1^*, \dots, v_n^*\}$ be a basis for $\mathcal{V}^*$, where $\mathcal{V}$ is a vector space over $\mathbb{F}$. Consider the alternating bilinear forms

$$(v_i^* \wedge v_j^*)(u_1, u_2) = (u_1, v_i^*)(u_2, v_j^*) - (u_1, v_j^*)(u_2, v_i^*), \qquad i < j.$$

Show that they are a basis for the space of alternating bilinear forms in $\mathcal{V}$.

2. (15 points) Let $S, T \in \mathcal{L}(\mathcal{V})$ be such that $ST = TS$, where $\mathcal{V}$ is a vector space over $\mathbb{C}$. Assume that $0 \neq v \in \mathcal{V}$ satisfies $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$.

(a) Show that $\mathcal{W} = \ker(T - \lambda I)$ is $S-$invariant and has dimension at least one.

(b) Consider the operator $S_{\mathcal{W}} \in \mathcal{L}(\mathcal{W})$ given by $\mathcal{S}_{\mathcal{W}} w = Sw$ for $w \in \mathcal{W}$. Show that is has at least one eigenvector.

(c) Conclude from the previous argument that two commuting operators in a $\mathbb{C}$-vector space have at least one common eigenvector.

(d) Find an example of the above where the eigenvalue is different.

3. (15 points) Let $T \in \mathcal{L}(\mathcal{M}(n, \mathbb{R}))$ be given by $TA = A + A^{Tr}$.

   (a) Compute the characteristic polynomial of T. (Hint. Use a basis of $\mathcal{M}(n, \mathbb{F})$ using exclusively symmetric and skew-symmetric matrices.)

   (b) Find $\min\mathrm{P}_{T,A}$ for an arbitrary A. (Hint. There are three cases: A symmetric, A skewsymmetric, and A neither symmetric nor skew-symmetric.)

   (c) Find $\min\mathrm{P}_T$.

   (d) Show that T is diagonalizable.

4. (15 points) Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ be a bilinear map and let $v^* \in \mathcal{V}^*$.

   (a) Show that $\psi(v_1, v_2) = (\Psi(v_1, v_2), v^*)$ defines a bilinear form.

   (b) Apply this to show that $\psi(A, B) = \mathrm{trace}(B^{Tr} A)$ is a bilinear form in $\mathcal{M}(n, \mathbb{R})$.

   (c) Show that ψ is symmetric.

5. (15 points) Let $T \in \mathcal{L}(\mathcal{V})$ be represented by the matrix

$$A = \begin{bmatrix} c & 1 & & \\ & c & 1 & \\ & & c & 1 \\ & & & c \end{bmatrix}.$$

(The numbers that are not displayed are zeros.) Show that it can also be represented by the matrix

$$B = \begin{bmatrix} c & & & \\ 1 & c & & \\ & 1 & c & \\ & & 1 & c \end{bmatrix}.$$

In other words, show that A and B are similar.

6. (15 points) An operator $T \in \mathcal{L}(\mathcal{V})$ is cyclic if there exists a basis $\{v_1, v_2, \ldots, v_n\}$ such that the associated matrix is

$$\begin{bmatrix} 0 & 0 & \ldots & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 0 & 1 & -a_{n-1} \end{bmatrix}.$$

In this case, the vector $v = v_1$ is called cyclic as well. Show that $\min\mathrm{P}_{T,v} = \chi_T$. (Hint. Write what having this matrix representation means in terms of the vectors of the basis.) Use this to prove that T is cyclic if and only if $\chi_T = \min\mathrm{P}_T$.

MATH 672: Vector spaces

Fall'13 — First midterm exam Part 2/2 — November 15

1. Let $T \in \mathcal{L}(\mathcal{V})$ be such that $\{v, Tv, T^2v, T^3\}$ is a basis for $\mathcal{V}$ and
$$\chi_T(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2 + \lambda + 1.$$
   (a) Compute T^4v.
   (b) What is the matrix representation for T in the given basis? (Hint. Write $v_1 = v, v_2 = Tv, ...$)
   (c) Compute $\text{minP}_{T,v}$.
   (d) Compute minP_T.

2. Consider the following functional $\rho \in \mathcal{M}(2, \mathbb{R})^*$
$$(A, \rho) = a_{12} - a_{21}.$$
Show that it is a basis for the annihilator of the set of symmetric matrices.

3. Let $T \in \mathcal{L}(\mathcal{V})$ be such that its matrix representation with respect to a basis $\{v_1, v_2, \dots, v_6\}$ is
$$\begin{bmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & 1 & 3 & \\ & & & & 1 & 3 \end{bmatrix}.$$
Compute all eigenvalues of T. Compute $\dim\ker(T - \lambda I)$ for arbitrary $\lambda \in \mathbb{R}$. Compute minP_{T,v_3}.

4. Let $T \in \mathcal{L}(\mathcal{V})$, $\{v_1, \dots, v_n\}$ be a basis for $\mathcal{V}$ and $P_j = \text{minP}_{T,v_j}$. Let then
$$P = \text{l.c.m.}\{P_1, \dots, P_n\}.$$
Then $P(T)(\sum_j c_j v_j) = 0$ for all $c_j \in \mathbb{F}$.

   This proves that P is a multiple of $P_{T,v}$ for all $v \in \mathcal{V}$.

   However, by definition ....

5. Let $A \in \mathcal{M}(n, \mathbb{R})$ be neither zero not the identity matrix. Assume that $A^2 = A$. What is minP_A? If $n = 4$, what are the possible characteristic polynomials of A.

6. Define:
   - Linear operator between two spaces
   - The dual space of a given vector space

- Annihilator of a set $A \subset \mathcal{V}$, where $\mathcal{V}$ is a vector space.

MATH 672: Vector spaces

Fall'13 | Quiz # 7 (Part A) | November 8

Plan B students, solve problems 1 and 2B.

1. (10 points) Let $T \in \mathcal{L}(\mathcal{V})$ (no finite dimension is assumed), let $\lambda_1, \lambda_2, \lambda_3$ be three pairwise different scalars and assume that

$$v_j \in \ker(T - \lambda_j I), \qquad j = 1, 2, 3.$$

Then

$$(T - \lambda_j I)v_i = (\lambda_i - \lambda_j)v_i.$$

Why?

If

$$v_1 + v_2 + v_3 = 0,$$

then

$$(\lambda_2 - \lambda_1)v_2 + (\lambda_3 - \lambda_1)v_3 = 0.$$

Why?

This implies that

$$(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)v_3 = 0.$$

Why?

Finally, we can prove that $v_3 = 0$, $v_2 = 0$, and $v_1 = 0$. How?

This proves that the subspaces $\ker(T-\lambda_1 I)$, $\ker(T-\lambda_2 I)$, and $\ker(T-\lambda_3 I)$ are independent.

2. (10 points) Let $S, T \in \mathcal{L}(\mathcal{V})$ be such that

$$T = R^{-1}SR, \qquad \text{with } R \in \mathbf{GL}(\mathcal{V}).$$

(a) What do we call S and T when this happens? How do we call $\mathbf{GL}(\mathcal{V})$?

(b) Show that

$$T^k = R^{-1}S^k R.$$

(c) Show that if $P \in \mathbb{F}[x]$, then

$$P(T) = R^{-1}P(S)R.$$

2B. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Compute its characteristic polynomial, all its eigenvalues and all its eigenvectors.

MATH 672: Vector spaces

Fall'13 | Quiz # 7 (Part B) | Due November 11

Plan B students, solve problems 3B, 4B, and 5B.

3. (5 points) Let T be a diagonalizable operator such that

$$\chi_T(\lambda) = (\lambda_1 - \lambda)^{n_1} \dots (\lambda_k - \lambda)^{n_k},$$

where $\lambda_1, \dots, \lambda_k$ are pairwise different and $n_1 + \dots + n_k = \dim \mathcal{V}$. What is minP_T? (Hint. T is diagonalizable!)

4. (5 points) Let $T \in \mathcal{L}(\mathcal{V})$, and assume that $\dim\ker(T - c\,I) = k$. Show that $(c - \lambda)^k$ is a factor in $\chi_T(\lambda)$. (Hint. Take k linearly independent eigenvectors, start a basis of $\mathcal{V}$ with them, and then look at the associated matrix.)

5. (10 points) Let $T = R^{-1}SR$, where $R \in \mathbf{GL}(\mathcal{V})$. Show that

$$\mathrm{minP}_{T,v} = \mathrm{minP}_{S,Rv}.$$

Use this to prove that $\mathrm{minP}_T = \mathrm{minP}_S$. (Hint. Use the property $P(T) = R^{-1}P(S)R$ that we proved in the in-class part of the exam.)

6. (10 points) We have used the fact that the characteristic polynomial of

$$\begin{bmatrix} 0 & 0 & \dots & \dots & 0 & -a_0 \\ 1 & 0 & \dots & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \dots & 0 & 1 & -a_{n-1} \end{bmatrix}$$

is $(-1)^n(\lambda^n + \sum_j a_j \lambda^j)$. Prove it. (Hint. Use the cofactor formula based on the last column of the matrix.)

3B. (10 points) Consider the following basis of $\mathbb{R}_2[x]$:

$$\mathbf{p} = \{1, x - \tfrac{1}{2}, x^2 - x + \tfrac{1}{6}\}.$$

Consider now the functionals

$$(p, \psi_0) = \int_0^1 p(x)dx, \qquad (p, \psi_1) = \int_0^1 p'(x)dx, \qquad (p, \psi_2) = \tfrac{1}{2}\int_0^1 p''(x)dx.$$

Show that $\{\psi_0, \psi_1, \psi_2\}$ is the dual basis to $\mathbf{p}$.

4B. (15 points) Consider an operator $T \in \mathcal{L}(\mathcal{V})$, and a basis $\{v_1, v_2, v_3\}$ of the space $\mathcal{V}$. Assume that

$$Tv_1 = c\,v_1, \qquad Tv_2 = c\,v_2 + v_1, \qquad Tv_3 = c\,v_3 + v_2,$$

where $c \in \mathbb{F}$.

(a) Write the matrix associated to T in this basis.

(b) Compute $\det T$.

(c) Show that $T \in \mathbf{GL}(\mathcal{V})$ if and only if $c \neq 0$.

(d) Compute $\chi_T(\lambda)$.

(e) Compute all eigenvalues and eigenvectors of T.

(f) Use the definition of minimal polynomial to compute minP_{T,v_1}, minP_{T,v_2} and minP_{T,v_3}.

5B. (5 points) Consider the following subspaces of $\mathbb{R}^3_r$:

$$\mathcal{V}_1 = \{[x, y, z] \,:\, x = y\},$$
$$\mathcal{V}_2 = \{[x, y, z] \,:\, x + y + z = 0\}.$$

Finda bases for $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{V}_1 \cap \mathcal{V}_2$. Show that the sum of $\mathcal{V}_1$ and $\mathcal{V}_2$ is not direct. Find a basis for $\mathcal{V}_1 + \mathcal{V}_2$.

MATH 672: Vector spaces

Fall'13 — Quiz # 6 (Part A) — November 1

Plan B students, solve problem 1, 2B and 3B.

1. (10 points) Justify all the steps of the following proof. Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, let $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_m\}$ be respective bases for $\mathcal{V}$ and $\mathcal{W}$, and assume we have the matrix representation

$$Tv_j = \sum_i a_{ij} w_i.$$

Let now

$$T^* w_j^* = \sum_i b_{ij} v_i^*$$

where $\{v_1^*, \ldots, v_n^*\}$ is the dual basis of $\mathbf{v}$, and $\{w_1^*, \ldots, w_m^*\}$ is the dual basis of $\mathbf{w}$. Then:

$$\begin{aligned}(Tv_j, w_i*) &= \big(\sum_l a_{lj} w_l, w_i^*\big) \\ &= \sum_l a_{lj} (w_l, w_i^*) \qquad \text{(Why?)} \\ &= a_{ij} \qquad \text{(Why?)}\end{aligned}$$

and

$$\begin{aligned}(v_j, T^* w_i^*) &= \big(v_j, \sum_k b_{ki} v_k^*\big) \\ &= \sum_k b_{ki} (v_j, v_k^*) \qquad \text{(Why?)} \\ &= b_{ji} \qquad \text{(Why?)}\end{aligned}$$

which proves that $A^{Tr} = B$. (Why?)

2. (10 points) Let $T \in \mathcal{L}(\mathcal{V})$ and $\{v_1, \dots, v_n\}$ be a basis for $\mathcal{V}$ such that

$$T v_j = c_j\, v_j \qquad \forall j, \qquad \text{with } c_j \in \mathbb{F}.$$

Using the definition, show that $\det T = c_1 \dots c_n$.

3. (10 points) Let $v^*, w^* \in \mathcal{V}^*$. Show that

$$(v^* \wedge w^*)(v_1, v_2) = (v_1, v^*)(v_2, w^*) - (v_1, w^*)(v_2, v^*)$$

is an alternating bilinear form in $\mathcal{V}$. (Hint. There are two things to prove.)

2B. (10 points) Find a basis for the subspace $\{p \in \mathbb{R}_2[x] \,:\, p(0) = p(1) = 0\}$.

3B. (10 points) Let $T \in \mathcal{L}(\mathbb{R}_2[x])$ be given by $Tp(x) = xp'(x)$. Write its coordinate matrix with respect to the basis $\{1, 1+x, 1+x^2\}$.

MATH 672: Vector spaces

Fall'13 — Quiz # 6 (Part B) — due November 4

Plan B students, solve problems 4, 5B and 6B.

4. (10 points) Consider the functionals ϕ_j given by $(p, \phi_j) = p(j)$. It is easy to show that $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ is a basis for $\mathbb{R}_3[x]^*$. (Don't prove it.) Using this basis, find a basis for $A^\perp$, where $A = \{1, x^2\}$.

5. (5 points) Let $T \in \mathcal{L}(\mathcal{V})$ and let $\{v_1, \dots, v_n\}$ be a basis for $\mathcal{V}$. Show that $T \in \mathbf{GL}(\mathcal{V})$ if and only if $\{Tv_1, \dots, Tv_n\}$ is a basis for $\mathcal{V}$. (You are allowed to use basic results on bases, dimensions, and the relation between the dimensions of kernel and range.)

6. (5 points) Let $T \in \mathcal{L}(\mathcal{V})$ and let $\{v_1, \dots, v_n\}$ be a basis for $\mathcal{V}$ such that

$$T v_j = \sum_{i=j}^{n} a_{ij} v_i.$$

Show that $\det T = a_{11} a_{22} \dots a_{nn}$. (Try to write a clean proof. You might need an induction argument.)

5B. (5 points) Let

$$S = \{[x, y, z] \, : \, x + y + z = 0\}, \qquad T = \{[x, y, z] \, : \, x - y = y - z = 0\}.$$

Show that $S \oplus T = \mathbb{R}^3_r$. (Hint. Check that the sum is direct and find the dimensions of S and T.)

6B. (5 points) Let $T \in \mathcal{L}(\mathcal{V})$ satisfy

$$Tv_j = v_{j+1}, \qquad j = 1, \ldots, n-1, \qquad Tv_n = -\sum_{i=1}^{n-1} a_{i-1} v_i,$$

for a given basis $\{v_1, \ldots, v_n\}$. Write down the matrix associated to T in this basis. Show that $T \in \mathbf{GL}(\mathcal{V})$ if and only if $a_0 \neq 0$. (Hint. Study the invertibility of the matrix instead.)

MATH 672: Vector spaces

Fall'13 — Quiz # 5 (Part A) — October 25

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1. (10 points) Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be an isomorphism and consider the matrix representations

$$Tv_j = \sum_i a_{ij} w_i, \qquad T^{-1} w_j = \sum_i b_{ij} v_i,$$

where $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_n\}$ are respective bases for $\mathcal{V}$ and $\mathcal{W}$. Here's an unfinished diagram

$$\begin{array}{ccccccc}
\mathcal{W} & \xrightarrow{T^{-1}} & \mathcal{V} & \xrightarrow{T} & \mathcal{W} & \xrightarrow{T^{-1}} & \mathcal{V} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}^n_c & \longrightarrow & \mathbb{F}^n_c & \longrightarrow & \mathbb{F}^n_c & \longrightarrow & \mathbb{F}^n_c
\end{array}$$

- With our usual notation ($C_{\mathbf{v}}$ is the coordinate map for the basis $\mathbf{v}$, A is the operator corresponding to multiplication by the matrix A,...), tag all the untagged arrows in the diagram.
- Find where the identity operators $I_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ and $I_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$ are in the diagram.
- Locate two multiplication operators by the identity matrix.

Finally, show that $B = A^{-1}$.

2. (10 points) Let $T \in \mathcal{L}(\mathcal{V})$ be such that there is a basis $\{v_1, v_2, \ldots, v_n\}$ with the property

$$Tv_j = j\, v_j \qquad j = 1, \ldots, n.$$

Write down the matrix associated to T in this basis.

3. (10 points) Let $\{v_1, \dots, v_n\}$ be a basis for the vector space $\mathcal{V}$, and let $\{v_1^*, \dots, v_n^*\}$ be the linear functionals characterized by

$$(v_i, v_j^*) = \delta_{ij}.$$

(a) Show that $v = \sum_j (v, v_j^*) v_j$ for all $v \in \mathcal{V}$. (Do not use any other characterization of the functionals v_j^*. Use only what is given to you.)

(b) Show that $\{v_1^*, \dots, v_n^*\}$ are linearly independent in $\mathcal{V}^*$.

MATH 672: Vector spaces

Fall'13 | Quiz # 5 (Part B) | Due October 28

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4. (10 points) Consider the following three functionals

$$(p, \phi_1) = p(1), \qquad (p, \phi_{-1}) = p(-1), \qquad (p, \phi_0^{(1)}) = p'(0),$$

defined in the space of all polynomials.

(a) Show that they are linearly dependent in $\mathbb{R}_2[x]^*$. (Hint. Make them act on a basis to see that there's a linear combination of them that vanishes on all polynomials.)

(b) Are they linearly independent in $\mathbb{R}_3[x]^*$?

5. (10 points) Consider the following subspaces of $\mathcal{M}(n;\mathbb{R})$:

$$S = \{A \in \mathcal{M}(n;\mathbb{R}) \,:\, A^{Tr} = A\}, \qquad T = \{A \in \mathcal{M}(n;\mathbb{R}) \,:\, A^{Tr} = -A\}.$$

(a) Show that $S \oplus T = \mathcal{M}(n;\mathbb{R})$. (Hint. Show first that the sum is direct. Then show that any matrix can be easily decomposed as the sum of a symmetric and a skewsymmetric matrix. Advise. Work with the matrix, not with its elements.)

(b) What are the dimensions of S and T? (Hint. $1 + 2 + \ldots + p = \frac{1}{2}p(p+1)$.)

MATH 672: Vector spaces

Fall'13 | Quiz # 4 | October 11

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1. (10 points) Consider the map

$$\begin{aligned} T : \mathbb{R}_3[x] &\longrightarrow \mathbb{R}_3[x] \\ p(x) &\longmapsto p(x) + p'(x). \end{aligned}$$

Show that it is linear.

Write its associated matrix with respect to the canonical basis $\{1, x, x^2, x^3\}$ of $\mathbb{R}_3[x]$.

2. (15 points) Consider the following two bases of $\mathbb{R}^3_r$:

$$\mathbf{v} = \{[1,1,1],[1,-2,3],[0,1,0]\}, \qquad \mathbf{e} = \{[1,0,0],[0,1,0],[0,0,1]\}.$$

Write the matrix for the change of basis from $\mathbf{v}$ to $\mathbf{e}$. (Careful! It does not really matter whether vectors are rows, columns, polynomials, etc. We always work on the premise that the matrix representation will lead to matrix-vector multiplication operators.)

Compute (or sketch the computation) the matrix for the change of variables from $\mathbf{e}$ to $\mathbf{v}$.

3. (15 points) Let $S, T \in \mathcal{L}(\mathcal{V})$, and let $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_n\}$ be two bases for $\mathcal{V}$. We consider the associated matrices defined by the relations

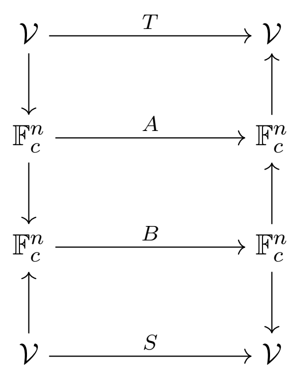
$$Tv_j = \sum_i a_{ij} v_i, \qquad Sw_j = \sum_i b_{ij} w_i,$$

and we assume that there exists $P \in \mathbf{GL}(n; \mathbb{F})$ such that

$$P^{-1}BP = A.$$

As usual, we identify a general matrix C with the operator $x \mapsto Cx$.

Fill the missing arrows in the following (commutative) diagram:



(Hint. you need to use the coordinate maps $C_{\mathbf{v}}$ and $C_{\mathbf{w}}$, multiplication by P, and the inverse of all of these operators.)

Consider the operator $R = C_{\mathbf{w}}^{-1} P C_{\mathbf{v}}$. Show that $R \in \mathbf{GL}(\mathcal{V})$. Identify R in the diagram.

Finally, show that $SR = RT$.

This proves that the operators S and T are similar.

MATH 672: Vector spaces

Fall'13 | First midterm exam Part 2/2 | October 14

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1. (15 points) Define:

   (a) Subspace of a vector space.

(b) Linear operator between two vector spaces.

(c) Basis of a (not necessarily finite dimensional) vector space.

2. (15 points) Let $S, T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, and let $\{v_1, \ldots, v_n\}$ be a basis for $\mathcal{V}$. Show that if

$$Sv_j = Tv_j \qquad \forall j,$$

then $Sv = Tv$ for all $v \in \mathcal{V}$.

3. (15 points) Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of an $n-$dimensional space $\mathcal{V}$, and assume that

$$\dim \mathcal{U} + \dim \mathcal{W} > n.$$

Let $\{u_j\}_{j=1}^{l}$ be a basis for $\mathcal{U}$ and $\{w_j\}_{j=1}^{m}$ be a basis for $\mathcal{W}$. The set

$$\{u_j\}_{j=1}^{l} \cup \{w_j\}_{j=1}^{m}$$

is linearly dependent. Why? (Hint. Count!)

Therefore there exists a linear combination

$$\sum_j c_j u_j + \sum_j d_j w_j = 0$$

with not all coefficients vanishing. If all the coefficients c_j are zero, we reach a contradiction. What contradiction?

Similarly, all the coefficients d_j cannot be zero. Why?

Therefore

$$v = \sum_j c_j u_j = -\sum_j d_j w_j$$

is a non-zero vector in $\mathcal{U} \cap \mathcal{W}$.Why?

4. (15 points) Show that

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

is a basis for $\mathcal{M}(2; \mathbb{R})$. (Hint. There's a simpler problem in $\mathbb{R}^4$ that you should solve instead.)

5. (15 points) Consider the linear map $T : \mathcal{M}(2; \mathbb{R}) \to \mathcal{M}(2; \mathbb{R})$ given by $TA = A + A^{Tr}$ (where the symbol A^{Tr} denotes the transposed matrix of A.

(a) Write down the coordinate matrix for the operator T in canonical basis of $\mathcal{M}(2; \mathbb{R})$.

(b) Repeat the exercise with the following basis of $\mathcal{M}(2; \mathbb{R})$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

6. (15 points) Consider the following two bases of $\mathbb{R}_1[x]$: $\mathbf{v} = \{1+x, 1-x\}$ and $\mathbf{w} = \{x, 2+x\}$. Let

$$p = 2(1+x) + 3(1-x) = -\tfrac{7}{2}x + \tfrac{5}{2}(2+x).$$

(a) What are $C_{\mathbf{v}}p$ and $C_{\mathbf{w}}p$?

(b) What is $C_{\mathbf{v}}^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$?

(c) Find the matrix that represents the change of basis $C_{\mathbf{w}}C_{\mathbf{v}}^{-1}$, from $\mathbf{v}$ and $\mathbf{w}$. (Hint. Using an intermediate basis simplifies the problem.)

(d) Check the result of (c) using the polynomial p.

MATH 672: Vector spaces

Fall'13 | First midterm exam Part 1/2 | Due October 9

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1. (20 points) Give a basis for the space $\mathbb{C}^3$ considered as a vector space over $\mathbb{R}$. (Prove that it is a basis: do not assume that you know the dimension in advance.)

2. (20 points) Let $\mathcal{U}, \mathcal{W}$ be subspaces of a vector space $\mathcal{V}$, with $\mathcal{U} \cap \mathcal{W} = \{0\}$. Assume that $\{u_1, \ldots, u_k\} \subset \mathcal{U}$ and $\{w_1, \ldots, w_l\} \subset \mathcal{W}$ are linearly independent sets. Prove that $\{u_1, \ldots, u_k\} \cup \{w_1, \ldots, w_l\}$ is linearly independent.

3. (20 points) Let $T \in \mathcal{L}(\mathcal{V})$ and consider the set

$$\mathrm{comm}[T] = \{S \in \mathcal{L}(\mathcal{V}) \,:\, ST = TS\}.$$

Show that comm$[T]$ is closed by addition, scalar multiplication, and operator multiplication.

4. (20 points) Show that if $B \in \mathcal{M}(m, n; \mathbb{F})$, then, the operator

$$\begin{array}{rcl} \mathcal{M}(n, k; \mathbb{F}) & \longrightarrow & \mathcal{M}(m, k; \mathbb{F}) \\ A & \longmapsto & BA \end{array}$$

is linear.

5. (20 points) Consider the following set of matrices

$$\mathcal{A} := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \,:\, n \in \mathbb{Z} \right\}.$$

Show that matrix multiplication makes it a group. Show that the operator $\mathbb{Z} \to \mathcal{A}$ assigning $n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is an isomorphism between the Abelian group $(\mathbb{Z}, +)$ and the Abelian group $(\mathcal{A}, \cdot)$.

6. (40 points) Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional spaces over $\mathbb{F}$, and let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

(a) Show that the set

$$\ker T = \{v \in \mathcal{V} \,:\, Tv = 0\}$$

is a subspace of $\mathcal{V}$.

(b) Show that the set

$$\operatorname{range} T = \{Tv \,:\, v \in \mathcal{V}\} = \{w \in \mathcal{W} \,:\, w = Tv \quad \text{for some} \quad v \in \mathcal{V}\}$$

is a subspace of $\mathcal{W}$.

(c) Show that if $\{v_1, \ldots, v_n\}$ is a basis for $\mathcal{V}$, then $\{Tv_1, \ldots, Tv_n\}$ is a spanning set for $\operatorname{range} T$.

(d) By completing a basis $\{\hat{v}_1, \ldots, \hat{v}_k\}$ of $\ker T$, we can build a basis of $\mathcal{V}$ with the form

$$\{v_1, \ldots, v_{n-k}, \quad \underbrace{\hat{v}_1, \ldots, \hat{v}_k}_{\text{basis of } \ker T} \quad \}.$$

Show that $\{Tv_1, \ldots, Tv_{n-k}\}$ is a basis for $\operatorname{range} T$. (Hint. The difficult part is showing linear independence. Note that if $T(\sum_j a_j v_j) = 0$, then $\sum_j a_j v_j \in \ker T = \operatorname{span}[\hat{v}_1, \ldots, \hat{v}_k]$.)

(e) Show that $\dim \ker T + \dim \operatorname{range} T = \dim \mathcal{V}$.

7. (20 points) Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional spaces over $\mathbb{F}$, and let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\widetilde{\mathbf{v}} = \{\widetilde{v}_1, \ldots, \widetilde{v}_n\}$ be two bases for $\mathcal{V}$, and let $\mathbf{w} = \{w_1, \ldots, w_m\}$ and $\widetilde{\mathbf{w}} = \{\widetilde{w}_1, \ldots, \widetilde{w}_m\}$ be bases for $\mathcal{W}$. Let now A be the matrix associated to T in the bases $\mathbf{v}$ and $\mathbf{w}$, and let B be the matrix associated to T in the bases $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{w}}$. Show that we can write

$$A = PBQ, \qquad P \in \boldsymbol{GL}(m, \mathbb{F}), \qquad Q \in \boldsymbol{GL}(n, \mathbb{F}).$$

Explain what the matrices P and Q are in terms of changes of bases.

MATH 672: Vector spaces

Fall'13 | Quiz # 3 | September 27

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1. (10 points) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the field $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be respective bases of $\mathcal{V}$ and $\mathcal{W}$.

1. Given $S, T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $c, d \in \mathbb{F}$, we can find coefficients $a_{ij}, b_{ij} \in \mathbb{F}$ such that

$$Sv_j = \sum_i a_{ij} w_i \quad \text{and} \quad Tv_j = \sum_i b_{ij} w_j$$

Why?

Therefore

$$\begin{aligned}(cS + dT)v_j &= c(Sv_j) + d(Tv_j) && \text{Why?}\\ &= c\sum_i a_{ij} w_i + d\sum_i b_{ij} w_j \\ &= \sum_i (c\, a_{ij} + d\, b_{ij}) w_i. && \text{Why?}\end{aligned}$$

This proves that, fixing bases of $\mathcal{V}$ and $\mathcal{W}$, the map $\Phi : \mathcal{L}(\mathcal{V}, \mathcal{W}) \to \mathcal{M}(m, n; \mathbb{F})$, associating linear operators to their matrices is ____________________.

2. Let now $(a_{ij}) \in \mathcal{M}(m, n; \mathbb{F})$. Then there exists a unique $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ such that

$$Tv_j = \sum_i a_{ij} w_i \qquad \forall j.$$

Why?

This proves that the map Φ is a ____________________, and therefore it is an ____________________ between $\mathcal{L}(\mathcal{V}, \mathcal{W})$ and $\mathcal{M}(m, n; \mathbb{F})$. Therefore the space $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is finite dimensional and its dimension is ____________________.

2. (5 points) Prove that if $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $S \in \mathcal{L}(\mathcal{W}, \mathcal{U})$, then $ST \in \mathcal{L}(\mathcal{V}, \mathcal{U})$.

3. (5 points) Consider the linear operator $T : \mathbb{R}_2[x] \to \mathbb{R}^3$ given by $Tp = [p(0), p(1), p(2)]$, and the basis $\{1, x, x^2\}$ of $\mathbb{R}_2[x]$ and canonical of $\mathbb{R}^3$. Write the matrix associated to T in these bases.

MATH 672: Vector spaces

Fall'13 | Quiz # 3 (Part B) | Due September 30

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1. (5 points) Show that if $\mathcal{V}$ is finite dimensional and $\mathcal{W}$ is a subspace of $\mathcal{V}$, then $\mathcal{W}$ is finite dimensional and $\dim\mathcal{W} \leq \dim\mathcal{V}$ with equality only if $\mathcal{W} = \mathcal{V}$. (Hint. In both cases, assume the opposite.)

2. (5 points) Show that $\mathcal{V}$ is finite dimensional and $\mathcal{W}$ is a subspace of $\mathcal{V}$, then the quotient space $\mathcal{V}/\mathcal{W}$ is finite dimensional. (Hint. It is easy to find a spanning set.)

3. (10 points) **A puzzle.** You are given a 7×6 matrix whose columns $c_1, \ldots, c_6 \in \mathbb{R}^7_c$ satisfy:
   - $c_1 \neq 0$
   - $c_2 = 3c_1$
   - c_3 is not a multiple of c_1
   - $c_4 = c_1 + 2c_2 + 3c_3$
   - $c_5 = 6c_3$
   - $c_6 \notin \text{span}[c_2, c_3]$

   What is the reduced row echelon form of the matrix? What is the rank of the matrix? What is a basis of the column space of the matrix? What is a basis of the row space of the matrix? (All these questions can be given specific answers up to a few numbers you will have to give a name to.)

MATH 672: Vector spaces

Fall'13 Quiz # 2 September 20

1. (10 points) Prove that in a finite dimensional vector space $\mathcal{V}$, a maximal linear independent set is spanning.
2. (10 points) Here is a proof. Justify every step (one or two well chosen words suffice):

   Let $\varphi : \mathcal{V} \to \mathcal{W}$ be an isomorphism between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ over the same field $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathcal{V}$ and let $w_j = \varphi(v_j)$ for $j = 1, \ldots, n$.
   If $\sum_j a_j w_j = \sum_j a_j \varphi(v_j) = 0$ for some scalars a_j, then $\varphi(\sum_j a_j v_j) = 0$,... (Why?)
   and therefore $\sum_j a_j v_j = 0$,... (Why?)
   which implies that $a_j = 0$ for all j. (Why?)
   Let now $w \in \mathcal{W}$. We can decompose $\varphi^{-1}(w) = \sum_j a_j v_j$ ... (Why?)
   and therefore $w = \sum_j a_j w_j$. (Why?)

   Can you state what we just proved as an assertion about the image of a basis under an isomorphism?
3. (10 points) Let $\mathcal{V}$ be a vector space and $A \subset \mathcal{V}$ be a possibly infinite subset of $\mathcal{V}$. Define what we understand by:
   (a) A is linearly independent.
   (b) A is spanning.
4. (10 points) Showing that $\{1 + x, 1 + x^2, x + x^2\}$ is a basis of $\mathbb{Q}_2[x]$ is equivalent to showing that some particular vectors form a basis of $\mathbb{Q}^3$. Can you say which and show that they are actually a basis of $\mathbb{Q}^3$?

MATH 672: Vector spaces

Fall'13 Quiz # 1 September 13

1. (10 points) Let $\mathcal{V}, \mathcal{W}$ be vector spaces over the same field $\mathbb{F}$ and let $\varphi : \mathcal{V} \to \mathcal{W}$ be a linear bijection. Show that the inverse map $\varphi^{-1} : \mathcal{W} \to \mathcal{V}$ is also linear.

   What is the name we give to a linear bijection?
2. (10 points) Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Explain the steps of the following proof (as well as what we intend to prove): for every $a \in \mathbb{F}$

   $a\,0 = a\,(0 + 0)$ (Why? What's 0 here?)

   and therefore $a\,0 = a0 + a0$ (Why?)

   and therefore $a0 = 0$. (Why?)

3. (10 points) If $*$ is a binary operation in a set G:

   (a) What do we understand by an identity element of G?

   (b) What do we mean when we say that $*$ is associative?

   (Wacht out! We do not assume commutativity. Also, be rigorous with the for all symbol.)

4. (10 points) Find all the solutions to the linear system

$$\begin{aligned} x + y + z - t &= 1, \\ x + y - z + t &= 1, \\ x + y \qquad\quad &= 1. \end{aligned}$$