NAME:

MATH 672: Vector spaces

Fall'13 | Final exam Part 1/2 | Partial solutions

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1. (15 points) Let $\mathcal{V}$ be a finite dimensional space and $T \in \mathcal{L}(\mathcal{V})$. Show that

$$\chi_T(0) = 0 \qquad \Longleftrightarrow \qquad \mathrm{minP}_T(0) = 0 \qquad \Longleftrightarrow \qquad \det T = 0.$$

Since $\chi_T(\lambda) = \det(T - \lambda I)$, it is clear that $\det T = \chi_T(0)$, which proves that

$$\chi_T(0) = 0 \qquad \Longleftrightarrow \qquad \det T = 0.$$

On the other hand $\chi_T(0) = 0$ if and only if $\lambda = 0$ is an eigenvalue of T, if and only if $\lambda = 0$ is a root of the minimal polynomial (all eigenvalues are roots of the minimal polynomial and viceversa).

2. (20 points) Consider the $n \times n$ matrix

$$A = \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}$$

(a) Compute minP_A.
A simple way consists of computing the powers $(A - \mu I)^k$ and see that they only vanish when $n = k$. Since $\chi_A(x) = (\mu - x)^n$, this proves that $\mathrm{minP}_A(x) = (x - \mu)^n$. In a different way, we can see that

$$(A - \mu I)e_1 = 0, \quad (A - \mu I)e_2 = e_1, \qquad (A - \mu I)e_n = e_{n-1}.$$

This proves that $\mathrm{minP}_{A,e_n}(x) = (x - \mu)^n$. (It has to be of the form $(x - \mu)^k$ because of the characteristic polynomial.) Since the minimal polynomial for A is the l.c.m. of all minimal polynomial for all vectors, $\mathrm{minP}_A(x) = (x - \mu)^n$.

(b) Find a vector $v \in \mathbb{F}_c^n$ such that $\mathrm{minP}_{A,v} = \mathrm{minP}_A$.
The vector $v = e_n$ satisfies this property as shown in (a).

(c) Assuming that $\mu \neq 0$, find an explicit formula for A^{-1} and justify it. (Hint. By checking the lowest dimensional cases, it is easy to guess a general formula.)
We can write

$$A^{-1} = \begin{bmatrix} \mu^{-1} & -\mu^{-2} & \mu^{-3} & \dots & (-1)^{n-1}\mu^{-n} \\ & \mu^{-1} & -\mu^{-2} & \ddots & \vdots \\ & & \mu^{-1} & \ddots & \mu^{-3} \\ & & & \ddots & -\mu^{-2} \\ & & & & \mu^{-1} \end{bmatrix}$$

This is better described with the matrix

$$E = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} = A - \mu I.$$

Then

$$A^{-1} = \mu^{-1} I - \mu^{-2} E + \mu^{-3} E^2 - \ldots + (-1)^{n-1} \mu^{-n} E^{n-1}.$$

The sum ends at E^{n-1}, since $E^n = 0$. But then

$$\begin{aligned} AA^{-1} &= (\mu I + E)(\mu^{-1} I - \mu^{-2} E + \mu^{-3} E^2 + \ldots + (-1)^{n-1} \mu^{-n} E^{n-1}) \\ &= I - \mu^{-1} E + \mu^{-2} E^2 - \ldots - (-1)^{n-1} \mu^{n-1} E^{n-1} \\ &\quad + \mu^{-1} E - \mu^{-2} E^2 + \ldots + (-1)^{n-1} \mu^{n-1} E^{n-1} \\ &= I. \end{aligned}$$

Another option is to check that $AA^{-1} = I$ by actually multiplying these matrices.

(d) Compute $\text{minP}_{A^{-1}}$.
From (c) it follows that $\chi_{A^{-1}}(x) = (\mu^{-1} - x)^n$. One easy way to see whi $\text{minP}_{A^{-1}}(x) = (x - \mu^{-1})^n$ and not a lower order power is to note that

$$A^{-1} - \mu^{-1} I = (I - \mu^{-1} A) A^{-1} = \mu^{-1} (\mu I - A) A^{-1}.$$

This clearly shows that $(A^{-1} - \mu^{-1} I)^k = 0$ if and only if $(A - \mu I)^k = 0$ which is true for $k \geq n$. Another option is to realize that e_n has an n degree minimal polynomial w.r.t. A^{-1}. Another option uses the formula for the inverse to notice that

$$A^{-1} - \mu^{-1} I = -\mu^{-2} E + \mu^{-3} E^2 - \ldots + (-1)^{n-1} \mu^{-n} E^{n-1}.$$

The $k-$th power of this matrix is a linear combination of powers of E starting at E^k. FOr $k < n$ this is a non-zero matrix. For $k \geq n$, this is zero.

3. (10 points) In $\mathbb{C}_c^n$, we consider the Euclidean (dot) inner product,

$$\langle u, v \rangle = \sum_j \overline{v}_j u_j.$$

Let now $q_1, \ldots, q_k \in \mathbb{C}_c^n$ and let $Q \in \mathcal{M}(n \times k; \mathbb{C})$ be the matrix whose columns are the vectors $\{q_j\}$.

(a) Show that $\{q_1, \ldots, q_k\}$ is an orthonormal set if and only if $Q^* Q = I_k$.
It is simple to see that if $A = Q^* Q$, then $a_{ij} = \langle q_j, q_i \rangle = q_i^* q_j$. The result is then obvious.

(b) If $k = n$, show that $\{q_1, \ldots, q_n\}$ is an orthonormal set if and only if $Q^* = Q^{-1}$.
By (a), the set is orthonomrmal if and only if $Q^* Q = I_n$. However, now Q is square. Therefore $Q^* Q = I_n$ if and only if Q is invertible and $Q^{-1} = Q^*$. (A square matrix is invertible if and only if it has a left or right inverse. That sided inverse is the inverse.)

4. (10 points) In $\mathbb{R}_3[x]$, we consider the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Let then

$$\mathcal{H}_1 = \{p \in \mathbb{R}_3[x] \,:\, p(1) = 0\}, \qquad \mathcal{H}_2 = \{p \in \mathbb{R}_3[x] \,:\, p'(1) = 0\}.$$

(a) Find an orthonormal basis for $\mathcal{H}_1 \cap \mathcal{H}_2$. (Show how to compute everything, but do not show all details in computation of integrals.)
A basis for $\mathcal{H}_1 \cap \mathcal{H}_2$ is

$$\{1 - 2x + x^2, 2 - 3x + x^3\}.$$

Another easy one is $\{(1-x)^2, (1-x)^3\}$. Let $p_1 = 1 - 2x + x^2$ and $p_2 = 2 - 3x + x^3$. Then we start the Gram-Schmodit process doing

$$\widetilde{q}_1 = p_1, \qquad \|\widetilde{q}_1\|^2 = \tfrac{1}{5}, \qquad \text{and we define } q_1 = \sqrt{5}\widetilde{q}_1.$$

We then define

$$\widetilde{q}_2 = p_2 - \langle p_2, q_1\rangle q_1 = p_2 - \langle p_2, \widetilde{q}_1\rangle 5\widetilde{q}_1 = p_2 - \tfrac{13}{20}p_1 = x^3 - \tfrac{13}{6}x^2 + \tfrac{4}{3}x - \tfrac{1}{6}.$$

Finally

$$\|\widetilde{q}_2\|^2 = \tfrac{1}{252} = \tfrac{1}{6\sqrt{7}} \qquad \text{and we define } q_2 = \sqrt{252}\,\widetilde{q}_2.$$

The $\{q_1, q_2\}$ is an orthonormal basis for $\mathcal{H}_1 \cap \mathcal{H}_2$.

(b) Find the orthogonal projection of $p(x) = x^2$ onto $\mathcal{H}^1 \cap \mathcal{H}^2$.
We just need to compute

$$\begin{aligned} \pi x^2 &= \langle x^2, q_1\rangle q_1 + \langle x^2, q_2\rangle q_2 \\ &= \langle x^2, \widetilde{q}^1\rangle 5\widetilde{q}_1 + \langle x^2, \widetilde{q}_2, \rangle 252\, \widetilde{q}_2 \\ &= \tfrac{14}{5}x^3 + \tfrac{59}{10}x^2 - \tfrac{17}{5}x - \tfrac{3}{10}. \end{aligned}$$

5. (15 points) Let $\mathcal{H}$ be a finite dimensional inner product space.

(a) Give a direct proof of the following result:

$$\text{range}(T) = (\ker T^*)^\perp.$$

For any subspace $\mathcal{W}$, we can show that $\mathcal{W}^{\perp\perp} = \mathcal{W}$. Then we only need to show that

$$(\text{range}(T)^\perp = \ker T^*.$$

However,

$$\begin{aligned} T^*u = 0 &\iff \langle T^*u, v\rangle = 0 \quad \forall v \\ &\iff \langle u, Tv\rangle = 0 \quad \forall v \\ &\iff \langle u, w\rangle = 0 \quad \forall w \in \text{range}(T). \end{aligned}$$

(b) Show that if $\mathcal{W}$ is a subspace of $\mathcal{H}$, then

$$\dim\mathcal{W} + \dim\mathcal{W}^{\perp} = \dim\mathcal{H}.$$

This follows from the equality $\mathcal{W} \oplus \mathcal{W}^{\perp} = \mathcal{H}$. We first show that the sum is direct by showing that $\mathcal{W} \cup \mathcal{W}^{\perp} = \{0\}$, which follows from

$$u \in \mathcal{W} \cap \mathcal{W}^{\perp} \quad \Longrightarrow \quad u \perp u \quad \Longrightarrow \quad u = 0.$$

Next, if $u \in \mathcal{H}$ and $\pi_{\mathcal{W}} u$ is the orthogonal projection of u onto $\mathcal{W}$, then $u - \pi_{\mathcal{W}} u \in \mathcal{W}^{\perp}$, and this proves that $\mathcal{W} + \mathcal{W}^{\perp} = \mathcal{H}$.

(c) Show that

$$\rho(T) = \rho(T^*)$$

using the previous results.
This one is easy now:

$$\begin{aligned}
\rho(T) &= \dim \operatorname{range}(T) \\
&= \dim \mathcal{H} - \dim(\operatorname{range} T)^{\perp} && \text{(by (b))} \\
&= \dim \mathcal{H} - \dim \ker T^* && \text{(by (a))} \\
&= \dim \mathcal{H} - \nu(T^*) \\
&= \rho(T^*).
\end{aligned}$$

6. (15 points) In $\mathbb{R}_3[x]$, we consider the following functionals

$$(p, \psi_j) = \frac{1}{j!} p^{(j)}(1), \qquad j = 0, 1, 2, 3,$$

the canonical basis $\mathbf{e} = \{1, x, x^2, x^3\}$ and its dual basis $\mathbf{e}^* = \{e_0^*, e_1^*, e_2^*, e_3^*\}$.

(a) Show that $\mathbf{q}^* = \{\psi_0, \psi_1, \psi_2, \psi_3\}$ is a basis for $\mathbb{R}_3[x]^*$.
Let

$$p^* = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 + c_3\psi_3 = 0.$$

Then

$$\begin{aligned}
0 =& (1, p^*) &=& c_0 \\
0 =& (x, p^*) &=& c_0 + c_1 \\
0 =& (x^2, p^*) &=& c_0 + 2c_1 + c_2 \\
0 =& (x^3, p^*) &=& c_0 + 3c_1 + 3c_2 + c_3.
\end{aligned}$$

This implies that $c_0 = c_1 = c_2 = c_3 = 0$ and the set $\mathbf{q}^*$ is linearly independent. Since $\dim \mathbb{R}_3[x]^* = \dim \mathbb{R}_3[x] = 4$, it is a basis.

(b) Write the matrix for the change of basis from $\mathbf{e}^*$ to $\mathbf{q}^*$.
For any functional p^*

$$p^* = \sum_j (e_j, p^*) e_j^* = (1, p^*)e_0^* + (x, p^*)e_1^* + (x^2, p^*)e_2^* + (x^3, p^*)e_3^*.$$

If we apply this to the elements of $\mathbf{q}^*$ we obtain the elements of the columns of the matrix for the change of variables from $\mathbf{q}^*$ to $\mathbf{e}^*$. These computations are implicit in (a). The matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Its inverse is the matrix that we are looking for:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

(c) Using the result of (b), find the coefficients of the decomposition

$$e_2^* = \sum_{j=0}^{3} c_j \psi_j.$$

These coefficients are in the third column of P in (b). Therefore $e_2 = \psi_2 - 3\psi_3$.

7. (15 points) In $\mathcal{M}(n;\mathbb{R})$ we consider the inner product

$$\langle A, B\rangle = \sum_{i,j} a_{ij} b_{ij} = \operatorname{trace}(B^{Tr} A)$$

and the operator $TA = A^{Tr}$.

(a) Show that T is self-adjoint.
This follows from this simple argument

$$\langle A, A^{Tr}\rangle = \sum_{i,j} a_{ij} a_{ji} = \sum_{i,j} a_{ji} a_{ij} = \langle A^{Tr}, A\rangle.$$

(b) Give a direct argument (no characteristic polynomial involved), to show that the only possible eigenvalues of T are ± 1.
Assume that $A^{Tr} = \lambda A$. Therefore

$$A = (A^{Tr})^{Tr} = (\lambda A)^{Tr} = \lambda A^{Tr} = \lambda^2 A$$

or $(\lambda^2 - 1)A = 0$. The only two possible roots are $\lambda = \pm 1$.

(c) Use these facts and the spectral theorem to prove that $\mathcal{M}(n;\mathbb{R}) = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$, where $\mathcal{H}_1$ is the set of symmetric matrices and $\mathcal{H}_{-1}$ is the set of skew-symmetric matrices, and that the sum is orthogonal.
The set $\ker(T + I)$ is the set of skew-symmetric matrices. The set $\ker(T - I)$ is the set of symmetric matrices. (This follows from the definition of T.) Eigenvectors for different eigenvalues are orthogonal because T is selfadjoint by (a). Also T is diagonalizable because it is selfadjoint, and its only eigenvalues are ± 1 by (b). Then, we can build an orthonormal basis of eigenvectors, that is, of symmetric and skew-symmetric matrices.

8. (20 points) Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator on a complex inner product space $\mathcal{H}$, and let $P \in \mathbb{R}[x]$. Show that $P(T)$ is also selfadjoint. If T is normal, is also $P(T)$ normal?

Let $P(x) = \sum_j a_j x^j$ with $a_j \in \mathbb{R}$. A simple computation (induction) shows that T^k is selfadjoint for all k. Then

$$\langle P(T)u, v\rangle = \sum_j a_j \langle T^j u, v\rangle = \sum_j a_j \langle u, T^j v\rangle = \langle u, \sum_j a_j T^j v\rangle = \langle u, P(T)v\rangle.$$

If T is normal, we can actually have complex coefficients and

$$T^* P(T) = T^* \sum_j a_j T^j = \sum_j a_j T^* T^j = \sum_j a_j T^j T^* = P(T)T^*,$$

where we have used that $T^* T^j = T^j T^*$, which can be proved by induction.

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MATH 672: Vector spaces

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1. (20 points) Define:

   (a) Subspace of a vector space

   It is a non-empty subset of the vector space that is closed by addition and scalar multiplication.

   (b) Linear operator between two vector spaces

   An operator $T : \mathcal{V} \to \mathcal{W}$, between two vector spaces over the same field, is linear when

   $$T(av_1 + bv_2) = aTv_1 + bTv_2 \qquad \forall v_1, v_2 \in \mathcal{V}, \quad \forall a, b \in \mathbb{F}.$$

   (c) Adjoint of a linear operator between two vector spaces
   Given a linear operator $T : \mathcal{V} \to \mathcal{W}$, its adjoint is the operator $T^* : \mathcal{W}^* \to \mathcal{V}^*$ such that

   $$(Tv, w^*) = (v, T^*w^*) \qquad \forall v \in \mathcal{V}, \qquad \forall w^* \in \mathcal{W}^*.$$

   (d) Adjoint of a linear operator in an inner product space

   The adjoint of $T : \mathcal{H} \to \mathcal{H}$ (where $\mathcal{H}$ is an inner product space equipped with an inner product $\langle \cdot, \cdot \rangle$) is the only operator $T^* : \mathcal{H} \to \mathcal{H}$ satisfying

   $$\langle Tu, v \rangle = \langle u, T^*v \rangle \qquad \forall u, v \in \mathcal{H}.$$

2. (20 points) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over $\mathbb{F}$ with $\dim\mathcal{V} = 4$ and $\dim\mathcal{W} = 6$. Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

   (a) What is $\nu(T^*) - \nu(T)$?
   Using the relation between rank and nullity and the fact that $\rho(T) = \rho(T^*)$, we prove that

   $$\begin{aligned}\nu(T^*) &= \dim \mathcal{W} - \rho(T^*) = 6 - \rho(T^*) \\ &= 6 - \rho(T) = 6 - (\dim \mathcal{V} - \nu(T)) = 2 + \nu(T),\end{aligned}$$

   so $\nu(T^*) - \nu(T) = 2$.

   (b) Show that T is not surjective.
   Since $\rho(T) = 4 - \nu(T) \leq 4$, it is not possible that $\rho(T) = 6$ and therefore T is not surjective.

   (c) Show that T^* is not injective.
   Since $\nu(T^*) = 6 - \rho(T^*) = 6 - \rho(T) \geq 2$ ($\rho(T) \leq \dim \mathcal{W} \leq 4$), it is clear that T^* has at least a two-dimensional kernel.

(d) If T is injective, what is its rank?
If $\nu(T)=0$, then $\rho(T)=\dim\mathcal{V}-\nu(T)=4$.

3. (25 points) Let $\mathcal{H}$ be a complex inner product space and $T\in\mathcal{L}(\mathcal{H})$.

(a) Show that T^*T is self-adjoint.
Directly, we can compute $(T^*T)^*=T^*T^{**}=T^*T$, using that $(AB)^*=B^*A^*$ and $A^{**}=A$. We can also note that

$$\langle (T^*T)^*u,v\rangle=\langle u,T^*Tv\rangle=\langle Tu,Tv\rangle=\langle T^*Tu,v\rangle \qquad \forall u,v\in\mathcal{H}.$$

(b) Show that $\ker(T^*T)=\ker T$. (Hint. $\langle T^*Tu,u\rangle=\langle Tu,Tu\rangle$.)
If $Tu=0$, then $T^*Tu=T^*0=0$. If $T^*Tu=0$, then

$$0=\langle T^*Tu,u\rangle=\langle Tu,Tu\rangle=\|Tu\|^2,$$

so $Tu=0$.

(c) Show that the operators $\frac{1}{2}(T+T^*)$ and $\frac{1}{2\imath}(T-T^*)$ are self-adjoint.
We can use the same arguments as in (a) plus $(aT)^*=\overline{a}T^*$. Also,

$$\langle \tfrac{1}{2}(T+T^*)u,v\rangle=\tfrac{1}{2}\langle u,(T^*+T)v\rangle=\langle u,\tfrac{1}{2}(T+T^*_v\rangle$$

and

$$\langle \tfrac{1}{2\imath}(T-T^*)u,v\rangle=\tfrac{1}{2\imath}\langle u,(T^*-T)v\rangle=\langle u,\tfrac{1}{\overline{2\imath}}(T^*-T)v\rangle=\langle u,\tfrac{1}{2\imath}(T-T^*)v\rangle$$

for all $u,v\in\mathcal{H}$.

4. (20 points) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be subspaces of a seven dimensional vector space $\mathcal{V}$ and assume that $\dim\mathcal{V}_1=3$ and $\dim\mathcal{V}_2=4$. Tabulate all possibilities of dimensions of $\mathcal{V}_1\cap\mathcal{V}_2$ and relate them to all possibilities of dimensions of $\mathcal{V}_1+\mathcal{V}_2$. Is the sum direct in any of the cases?

We use the formula

$$\dim(\mathcal{V}_1+\mathcal{V}_2)+\dim(\mathcal{V}_1\cap\mathcal{V}_2)=\dim\mathcal{V}_1+\dim\mathcal{V}_2=7.$$

and note that $\dim(V_1\cap\mathcal{V}_2)\leq\dim\mathcal{V}_1=3$. Therefore we can have

$\dim(\mathcal{V}_1+\mathcal{V}_2)$	$\dim(\mathcal{V}_1\cap\mathcal{V}_2)$	Notes
7	0	$\mathcal{V}_1\oplus\mathcal{V}_2=\mathcal{V}$
6	1	
5	2	
4	3	$\mathcal{V}_1\subset\mathcal{V}_2, \qquad \mathcal{V}_1+\mathcal{V}_2=\mathcal{V}_2$

5. (30 points) Let $T\in\mathcal{L}(\mathcal{V},\mathcal{W})$ be surjective and

$$\dim\mathcal{V}=n>m=\dim(\mathcal{W}).$$

Let $\mathbf{v}=\{v_1,\ldots,v_n\}$ be a basis of $\mathcal{V}$, where $\ker T=\operatorname{span}[v_{m+1},\ldots,v_n]$. Finally, let $\mathbf{w}=\{w_1,\ldots,w_m\}$ be given by

$$w_j=Tv_j, \qquad j=1,\ldots,m.$$

(a) Show that $\mathbf{w}$ is a basis for $\mathcal{W}$.
We only need to show linear independence. If $\sum_j c_j w_j = 0$, then using linearlity of T

$$0 = \sum_{j=1}^{m} c_j T v_j = T(\sum_{j=1}^{m} c_j v_j) \quad \Longrightarrow \quad \sum_{j=1}^{m} c_j v_j \in \ker T = \operatorname{span}[v_{m+1}, \ldots, v_n].$$

By linear independence of the vectors $\{v_1, \ldots, v_n\}$ it follows that $c_j = 0$ for all j.

(b) Consider now the operator $U \in \mathcal{L}(\mathcal{W}, \mathcal{V})$ given by

$$U w_j = v_j, \qquad j = 1, \ldots, m.$$

Show that $TUw = w$ for all $w \in \mathcal{W}$.
By definition of w_j and U

$$TUw_j = Tv_j = w_j \qquad j = 1, \ldots, m.$$

Since $\{w_1, \ldots, w_m\}$ is a basis of $\mathcal{W}$, it follows that

$$TUw = TU(\sum_j c_j w_j) = \sum_j c_j TU w_j = \sum_j c_j w_j = w.$$

(c) Show that U is injective but not surjective.
If $Uw = 0$, then $0 = TUw = w$, so $\ker U = \{0\}$. The operator U cannot be surjective because $\rho(U) = \dim \mathcal{W} - \nu(U) \leq m < n = \dim \mathcal{V}$,

(d) Compute UTv_j for all j. (Hint. There are two groups of v_j vectors.)
We have

$$UTv_j = Uw_j = v_j, \qquad j = 1, \ldots, m, \qquad \text{and} \qquad UTv_j = U0 = 0, \quad j = m{+}1, \ldots, n.$$

(e) Write down the matrix for T in the given bases.
Using (d), it follows that the matrix for T is

$$\begin{bmatrix} 1 & & & 0 & \ldots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & 1 & 0 & \ldots & 0 \end{bmatrix} = \begin{bmatrix} I_{m \times m} & 0_{m \times n-m} \end{bmatrix}$$

(f) Write down the matrix for U in the given bases.
Using (b), it is clear that the matrix is

$$\begin{bmatrix} I_{m \times m} \\ 0_{n-m \times m} \end{bmatrix}$$

6. (15 points) Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space $\mathcal{V}$ and let

$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \qquad j = 1, \ldots, n,$$

for some given coefficients $a_{ij} \in \mathbb{F}$. Show that $\{w_1, \ldots, w_n\}$ is a basis for $\mathcal{V}$ if and only if the matrix A is invertible.

Consider the linear operator $T \in \mathcal{L}(\mathcal{V})$ given by

$$Tv_j = w_j = \sum_i a_{ij} v_i.$$

Its associate matrix with respect to the given basis is $\mathcal{V}$ is A. Then T is invertible if and only if A is invertible. On the other hand, T is invertible if and only if the image of a basis is a basis. (This is easy to prove.)

7. (30 points) Let $\mathcal{H}$ be an inner product space, $\{q_1, \ldots, q_k, p_1, \ldots, p_m\}$ be an orthonormal basis and let

$$P_1 u = \sum_{j=1}^{k} \langle u, q_j \rangle q_j, \qquad P_2 u = \sum_{j=1}^{m} \langle u, p_j \rangle p_j.$$

(a) Show that the operators P_1 and P_2 are self-adjoint.
It has to be proved only for one of them, since both have the same structure. Then

$$\begin{aligned} \langle P_1 u, v \rangle &= \langle \sum_j \langle u, q_j \rangle q_j, v \rangle = \sum_j \langle u, q_j \rangle \langle q_j, v \rangle \\ &= \sum_j \langle u, q_j \rangle \overline{\langle v, q_j \rangle} = \langle u, \sum_j \langle v, q_j \rangle q_j \rangle = \langle u, P_1 v \rangle, \end{aligned}$$

after applying simple properties of the inner product (sesquilinearity). This proves that $P_1^* = P_1$.

(b) Show that $P_1 P_2 = 0$ and $P_2 P_1 = 0$.
By orthogonality of $\{q_1, \ldots, q_k, p_1, \ldots, p_m\}$ it follows that

$$P_1 p_j = 0, \qquad \forall j.$$

Since $\mathrm{range} P_2 = \mathrm{span}[p_1, \ldots, p_m]$, this implies that $P_1 P_2 = 0$. The other equality follows by reversing the roles of the vectors $\{p_i\}$ and $\{q_i\}$.

(c) Show that if $\lambda_1, \lambda_2 \in \mathbb{C}$, then the operator

$$T = \lambda_1 P_1 + \lambda_2 P_2.$$

is normal.
By (a), $T^* = \overline{\lambda_1} P_1 + + \overline{\lambda_2} P_2$. Then by (b)

$$T^* T = (\overline{\lambda_1} P_1 + \overline{\lambda_2} P_2)(\lambda_1 P_1 + \lambda_2 P_2) = |\lambda_1|^2 P_1^2 + |\lambda_2|^2 P_2^2 = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2.$$

In the last inequality we have used that $P_j^2 = P_j$, although this is not needed for the argument. Similarly we show that

$$TT^* = |\lambda_1|^2 P_1^2 + |\lambda_2|^2 P_2^2 = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2.$$

This proves that T is normal.

(d) Give necessary and sufficient conditions for T to be self-adjoint.
It is clear that if $\lambda_1, \lambda_2 \in \mathbb{R}$, then $T^* = T$. If T is selfadjoint, it can only have real eigenvalues. However, $Tq_j = \lambda_1 q_j$ and $Tp_j = \lambda_2 p_j$, so $\lambda_1, \lambda_2 \in \mathbb{R}$.

(e) Compute $\min P_T$.
The operator $\min P_T$ is diagonalizable. (Its matrix in the given basis is diagonal.) Therefore its minimal polynomial has only simple roots. If $\lambda_1 \neq \lambda_2$, then $\min P_T(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. If $\lambda_1 = \lambda_2$, then $\min P_T(\lambda) = \lambda - \lambda_1$ and actually $T = \lambda_1 I$.

(f) Compute $\det T$.
Looking at the diagonal matrix associated to T in the given basis, it is clear that $\det T = \lambda_1^k \lambda_2^m$.

8. (20 points) Let $T \in \mathcal{L}(\mathcal{V})$ be such that its matrix representation with respect to the basis $\{v_1, \ldots, v_6\}$ is

$$A = \begin{bmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & 1 & 3 & \\ & & & & 1 & 3 \end{bmatrix}$$

(a) Show that we cannot define an inner product in $\mathcal{V}$ such that T is self-adjoint.
There are several ways to show that T is not diagonalizable. The simplest way is by looking at $\rho(A - 2I)$ and $\rho(A_3 I)$. Both are 5, so T has only two linearly independent eigenvectors. If T were selfadjoint, it'd be diagonalizable, so it is not selfadjoint with respect to any inner product.

(b) Find a basis for $\mathcal{V}$ such that the associated matrix is

$$\begin{bmatrix} 3 & 1 & & & & \\ & 3 & 1 & & & \\ & & 3 & & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \\ & & & & & 2 \end{bmatrix}$$

We just need to reorder the basis: $\{v_6, v_5, v_4, v_1, v_2, v_3\}$.

(c) Assume that $P \in \mathbb{R}_7[x]$ satisfies $P(A) = 0$ and $P(4I) = 0$. What is P?
Some work shows that $\min P_T = \min P_A = (\lambda - 2)^3(\lambda - 3)^3$. For instance, $\min P_{T,v_3}(\lambda) = (\lambda - 2)^3$ and $\min P_{T,v_4}(\lambda) = (\lambda - 2)^3$. (This has to be proved.) Then if $P(A) = 0$, $P(\lambda) = Q(\lambda)\min P_T(\lambda)$. However, $P(4I) = 0$, which means that $P(4) = 0$. Therefore

$$P(\lambda) = a\,(\lambda - 4)(\lambda - 3)^3(\lambda - 2)^3, \qquad a \in \mathbb{C}.$$

(d) Show that $\{v_3, Tv_3, T^2 v_3, v_4, Tv_4, T^2 v_4\}$ is a basis for $\mathcal{V}$ and write down the matrix representing T with respect to this basis.
On the one hand

$$(T - 2I)v_3 = v_2, \qquad (T - 2I)^2 v_3 = (T - 2I)v_2 = v_1.$$

This implies that

$$\operatorname{span}[T, v_3] = \operatorname{span}[v_3, Tv_3, T^2 v_3] = \operatorname{span}[v_1, v_2, v_3].$$

Similarly $\mathrm{span}[v_4, Tv_4, T^2v_4] = \mathrm{span}[v_4, v_5, v_6]$. Therefore the sets $\{v_3, Tv_3, T^2v_3\}$ and $\{v_4, Tv_4, T^2v_4\}$ are linearly independent. Since their spans are independent subspaces (this is clear by looking at the second bases we are given for each of them), then the union of their bases is a basis for the sum. (There are many other ways of showing this.) Finally, we write

$$0 = (T-2I)^3v_3 = T^3v_3 - 6T^2v_3 + 12Tv_3 - 8v_3, \quad \text{that is} \quad T^3v_3 = 8v_3 - 12Tv_3 + 6T^2v_3.$$

With an identical argument

$$T^3v_4 = 27v_4 - 27Tv_4 + 9T^2v_4.$$

With this, the matrix with respect to this basis is

$$\begin{bmatrix} & & 8 & & & \\ 1 & & -12 & & & \\ & 1 & 6 & & & \\ & & & & & 27 \\ & & & 1 & & -27 \\ & & & & 1 & 9 \end{bmatrix}$$