NAME:

MATH 672: Vector spaces

Fall'13 | Second midterm exam Part 2/2 | Partial solutions

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1. (20 points) Define:

   (a) Linear functional on a vector space $\mathcal{V}$.
   It is a linear map from $\mathcal{V}$ to its underlying field $\mathbb{F}$.

   (b) The dual space of a vector space $\mathcal{V}$.
   It is the set of all linear functionals on $\mathcal{V}$, endowed with addition of operators and multiplication of operators by scalars/

   (c) Alternating $k-$linear form in a vector space $\mathcal{V}$.
   It is a map of k variables, $\Psi : \mathcal{V} \times \ldots \times \mathcal{V} \to \mathbb{F}$, which is linear in each variable and such that

   $$\Psi(v_{\sigma_1}, \ldots, v_{\sigma_k}) = \operatorname{sgn}(\sigma)\Psi(v_1, \ldots, v_k)$$

   for all $\sigma \in S_k$ and $v_1, \ldots, v_k \in \mathcal{V}$.

   (d) The transpose/adjoint of an operator $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.
   It is the operator $T^* : \mathcal{W}^* \to \mathcal{V}^*$ given by the relation

   $$(Tv, w^*) = (v, T^*w^*) \qquad \forall v \in \mathcal{V}, \quad w^* \in \mathcal{W}^*,$$

   or equivalently $T^*w^* = w^*T$.

2. (15 points) State (giving all hypotheses):

   (a) Cayley-Hamilton's Theorem.
   If T is a linear operator in a finite dimensional space and χ_T is its characteristic polynomial, then $\chi_T(T) = 0$.

   (b) The relation between the nullity and the rank of an operator $\mathcal{L}(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional.
   $\rho(T) + \nu(T) = \dim\mathcal{V}$.

   (c) The result that relates the dimensions of the intersection and the sum of two subspaces of a finite dimensional space.
   $\dim\mathcal{V}_1 \cap \mathcal{V}_2 + \dim\mathcal{V}_1 + \mathcal{V}_2 = \dim\mathcal{V}_1 + \dim\mathcal{V}_2$.

3. (25 points) Let T be given by the following relations

   $$Tv_1 = 2v_1, \qquad Tv_2 = 2v_2 + v_1, \qquad Tv_3 = -v_3, \qquad Tv_4 = -v_4 + v_3, \qquad Tv_5 = 2v_5,$$

   where $\{v_1, \ldots, v_5\}$ is a basis for $\mathcal{V}$.

   (a) Find a basis for $\mathcal{V}$ such that the associated matrix is

   $$A = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & 1 & 2 & & \\ & & & -1 & \\ & & & 1 & -1 \end{bmatrix}$$

(The numbers that are not displayed are zero.)

Let $\mathbf{w} = \{v_5, v_2, v_1, v_4, v_3\}$. It is simple to check that the matrix w.r.t. to $\mathbf{w}$ is A.

(b) Find all eigenvectors of A. Relate them to eigenvectors of T. (Recall that eigenvectors are organized in subspaces, so eigenvectors should be returned as spans.)
Note that $\chi_T(\lambda) = \chi_A(\lambda) = (2-\lambda)^3(-1-\lambda)^2$. Note also that

$$A - 2\mathrm{I} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & 1 & 0 & & \\ & & & -3 & \\ & & & 1 & -3 \end{bmatrix} \qquad A + I = \begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & 1 & 3 & & \\ & & & 0 & \\ & & & 1 & 0 \end{bmatrix}$$

If $\{e_1, \ldots, e_5\}$ are the canonical vectors of $\mathbb{F}^5$, then it is a simple observation to note that

$$\ker(A - 2I) = \operatorname{span}[e_1, e_3], \qquad \ker(A + I) = \operatorname{span}[e_5].$$

Therefore, relating coordinates in $\mathbb{F}^5$ to vectors of $\mathcal{V}$ represented in the basis of (a),

$$\ker(T - 2I) = \operatorname{span}[v_5, v_1], \qquad \ker(T + I) = \operatorname{span}[v_3].$$

Note that these vectors could already be observed in the original hypotheses of the problem, but we had to show that these were all the eigenvectors.

(c) Find the minimal polynomial for v_2 and v_5.
Since $Tv_5 = 2v_5$, then $\operatorname{minP}_{T,v_5}(\lambda) = \lambda - 2$. From the hypotheses it follows that

$$(T - 2I)v_2 = v_1, \qquad (T - 2I)^2 v_2 = (T - 2I)v_1 = 0,$$

so $\operatorname{minP}_{T,v_2}(\lambda) = (\lambda - 2)^2$.

(d) Give a reasoned guess of what the minimal polynomial of T is.
Apart from three vectors in the basis that are eigenvectors, we have $\operatorname{minP}_{T,v_2}(\lambda) = (\lambda - 2)^2$ and with exactly the same argument $\operatorname{minP}_{T,v_4}(\lambda) = (\lambda + 1)^2$. Taking the l.c.m. of the minimal polynomials for all the vectors in the basis, it follows that $\operatorname{minP}_T(\lambda) = (\lambda - 2)^2(\lambda + 1)^2$.

(e) Is T diagonalizable? Why/why not?
From (b) it is clear that we cannot find a basis of $\mathcal{V}$ using eigenvectors of T. Therefore T is not diagonalizable.

4. (20 points) Find the subspace $\mathcal{V} \subset \mathbb{R}_3[x]$ such that

$$\mathcal{V}^\perp = \operatorname{span}[\phi_0, \phi_1], \qquad (p, \phi_0) = p(0), \qquad (p, \phi_1) = p(1).$$

We are looking for all $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that

$$0 = (p, \phi_0) = a_0, \qquad 0 = (p, \phi_1) = a_0 + a_1 + a_2 + a_3.$$

This is clearly the same as $\operatorname{span}[-x + x^2, -x + x^3]$.

5. (20 points) Consider the operator $T : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = (1+x)p(x)$. Let $\mathbf{e} = \{1, x\}$ and $\mathbf{f} = \{1, x, x^2\}$ be the respective canonical bases for $\mathbb{R}_1[x]$ and $\mathbb{R}_2[x]$, and let $\mathbf{e}^*$ and $\mathbf{f}^*$ be the corresponding dual bases for $\mathbb{R}_1[x]^*$ and $\mathbb{R}_2[x]^*$. What is the matrix for T^* written in the bases $\mathbf{f}^*$ and $\mathbf{e}^*$?

Note that

$$T1 = 1 + x, \qquad Tx = x + x^2.$$

The matrix representation for T w.r.t. $\mathbf{e}$ and $\mathbf{f}$ is

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix represenation for T^* w.r.t. $\mathbf{f}^*$ and $\mathbf{e}^*$ is A^{Tr}.

6. (20 points) If $T = R^{-1}SR$, where $R \in \mathbf{GL}(\mathcal{V})$, prove that $P(T) = R^{-1}P(S)R$ for every polynomial $P \in \mathbb{F}[x]$. Conclude that the minimal polynomials for T and S are the same.

Note that $T^k = R^{-1}S^kR$ for all $k \geq 0$. (This is easy to prove by induction.) Then if $P(x) = \sum_j a_j x^j$

$$P(T) = \sum_j a_j T^j = \sum_j a_j R^{-1} S^j R = R^{-1}(\sum_j a_j S^j)R = R^{-1}P(S)R,$$

where we have used elementary properties of matrix multiplication. Therefore $P(T) = 0$ if and only if $R^{-1}P(S)R = 0$, and since R is invertible, if and only if $P(S) = 0$. Using the definition of minP it follows that the minimal polynomial for both operators is the same.

7. (20 points) Let $S \in \mathcal{L}(\mathcal{V})$. Then

$$\ker S^k \subseteq \ker S^{k+1} \qquad \forall k.$$

Why?

If $S^k v = 0$, then $S^{k+1}v = SS^k v = 0$.

Assume now that

$$\ker S^k = \ker S^{k+1},$$

pick $u \in \mathcal{V}$ such that $S^{k+2}u = 0$, and define $v = Su$. Then $v \in \ker S^{k+1}$. Why?

$S^{k+1}v = S^{k+2}u = 0$.

This implies that $S^{k+1}u = 0$. Why?

We have shown that $v \in \ker S^{k+1} = \ker S^k$, and therefore $S^k v = 0$, which implies that $S^{k+1}u = S^k v = 0$.

We have thus proved that if $\ker S^k = \ker S^{k+1}$, then $\ker S^{k+1} = \ker S^{k+2}$. Explain how.

The previous argument shows that $\ker S^{k+2} \subseteq \ker S^{k+1}$ for this particular value of k. However, $\ker S^{k+1} \subseteq \ker S^{k+2}$, and this proves that both sets are equal.

8. (20 points) Let A be a square matrix such that $A \neq 0$ and $A \neq I$. Assume that

$$A^2 - A = 0.$$

Show that

$$\chi_A(x) = (-1)^n x^k (x-1)^{n-k}, \qquad 1 \leq k \leq n-1.$$

The polynomial $P(\lambda) = \lambda^2 - \lambda$ satisfies $P(A) = 0$ and neither of its divisors satisfies this property. Therefore $\mathrm{minP}_A(\lambda) = \lambda(\lambda - 1)$. The characteristic polynomial is a multiple of this polynomial containing no other irreducible factors (roots). Therefore it has the given form.

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1. (15 points) Let $\{v_1^*, \ldots, v_n^*\}$ be a basis for $\mathcal{V}^*$, where $\mathcal{V}$ is a vector space over $\mathbb{F}$. Consider the alternating bilinear forms

$$(v_i^* \wedge v_j^*)(u_1, u_2) = (u_1, v_i^*)(u_2, v_j^*) - (u_1, v_j^*)(u_2, v_i^*), \qquad i < j.$$

Show that they are a basis for the space of alternating bilinear forms in $\mathcal{V}$.

Let $\{v_1, \ldots, v_n\}$ be the basis of $\mathcal{V}$ such that

$$(v_i, v_j^*) = \delta_{ij}. \tag{1}$$

We first show linear independence of the given bilinear forms. If $\sum_{i<j} a_{ij}(v_i^* \wedge v_j^*) = 0$, then for $l < m$

$$\begin{aligned}
0 &= \sum_{i<j} a_{ij}(v_i^* \wedge v_j^*)(v_l, v_m) \\
&= \sum_{i<j} a_{ij}((v_l, v_i^*)(v_m, v_j^*) - (v_l, v_j^*)(v_m, v_i^*)) \\
&= \sum_{i<j} a_{ij}(\delta_{l,i}\delta_{m,j} - \delta_{l,j}\delta_{m,i}) && \text{(by (1))} \\
&= a_{lm}. && (i < j,\ l < m \text{ so } i = m, j = l \text{ not possible})
\end{aligned}$$

This proves that the set is linearly independent. To prove that it is spanning, we take ψ bilinear and alternating, define $a_{ij} = \psi(v_i, v_j)$ for $i < j$ and

$$\varphi = \sum_{i<j} a_{ij}(v_i^* \wedge w_j^*).$$

With the same argument above, we verify that $\varphi(v_i, v_j) = a_{ij} = \psi(v_i, v_j)$ for all $i < j$. However, both bilinear forms are alternating, so this proves that $\varphi(v_i, v_j) = \psi(v_i, v_j)$ for all i, j.

Alternatively, we can count the number of elements $v_i^* \wedge v_j^*$ and notice that it coincides with the dimension of the subspace of alternating bilinear forms, which is $\frac{1}{2}n(n = 1)$.

2. (15 points) Let $S, T \in \mathcal{L}(\mathcal{V})$ be such that $ST = TS$, where $\mathcal{V}$ is a vector space over $\mathbb{C}$. Assume that $0 \neq v \in \mathcal{V}$ satisfies $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$.

   (a) Show that $\mathcal{W} = \ker(T - \lambda I)$ is $S-$invariant and has dimension at least one.
   By hypothesis $0 \neq v \in \mathcal{W}$, so $\mathcal{W}$ is at least one dimensional. If $w \in \mathcal{W}$, then

$$\begin{aligned}
(T - \lambda I)Sw &= S(T - \lambda I)w && (S \text{ and } T \text{ commute}) \\
&= S0 = 0 && (w \in \mathcal{W})
\end{aligned}$$

   which proves that $\mathcal{W}$ is S-invariant.

(b) Consider the operator $S_{\mathcal{W}} \in \mathcal{L}(\mathcal{W})$ given by $\mathcal{S}_{\mathcal{W}} w = Sw$ for $w \in \mathcal{W}$. Show that is has at least one eigenvector.

$S_{\mathcal{W}}$ is a linear operator in a vector space of dimension at least one over $\mathbb{C}$. Therefore its characteristic polynomial has at least one root μ, and there exists $0 \neq w \in \mathcal{W}$ such that $S_{\mathcal{W}} w = Sw = \mu w$.

(c) Conclude from the previous argument that two commuting operators in a $\mathbb{C}$-vector space have at least one common eigenvector.

The vector in (b) is an eigenvector for S, and for T, since it is an element of $\mathcal{W} = \ker(T - \lambda I)$.

(d) Find an example of the above where the eigenvalue is different.

Take $T = I$ and $S = 0$ in any $\mathcal{V}$. All vectors are eigenvectors and the eigenvalue is 1 for T and 0 for S.

3. (15 points) Let $T \in \mathcal{L}(\mathcal{M}(n, \mathbb{R}))$ be given by $TA = A + A^{Tr}$.

(a) Compute the characteristic polynomial of T. (Hint. Use a basis of $\mathcal{M}(n, \mathbb{F})$ using exclusively symmetric and skew-symmetric matrices.)

Let

$$\begin{aligned} \mathcal{V}_1 &= \{A \in \mathcal{M}(n, \mathbb{F}) \, : \, A^{Tr} = A\}, \\ \mathcal{V}_2 &= \{A \in \mathcal{M}(n, \mathbb{F}) \, : \, A^{Tr} = -A\}. \end{aligned}$$

It is simple to see that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{M}(n, \mathbb{F})$ and $\dim \mathcal{V}_1 = \frac{1}{2} n(n+1) = k$, while $\dim \mathcal{V}_2 = \frac{1}{2} n(n-1)$. Consider a basis of the space in the form

$$\{\underbrace{A_1, \ldots, A_k}_{\text{basis for } \mathcal{V}_1}, \underbrace{A_{k+1}, \ldots, A_{n^2}}_{\text{basis for } \mathcal{V}_2}\}.$$

Since $TA = 2A$ for $A \in \mathcal{V}_1$ and $TA = 0$ for $A \in \mathcal{V}_2$, the matrix representing T in this basis is diagonal

$$\begin{bmatrix} 2I_k & 0 \\ 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of T is the same as the one for the matrix, namely $(2 - \lambda)^k (-\lambda)^{n^2 - k}$ where $k = \frac{1}{2} n(n+1)$.

(b) Find $\min\mathrm{P}_{T,A}$ for an arbitrary A. (Hint. There are three cases: A symmetric, A skewsymmetric, and A neither symmetric nor skew-symmetric.)

If $0 \neq A \in \mathcal{V}_1$, then $(T - 2I)A = 0$ so $\min\mathrm{P}_{T,A}(\lambda) = \lambda - 2$. SImilarly for $0 \neq A \in \mathcal{V}_2$, $\min\mathrm{P}_{T,A} = \lambda$. Finally, for a general A,

$$TA = A + A^{Tr}, \qquad T^2 A = 2(A + A^{Tr}),$$

so $(T^2 - 2T)A = 0$. If $A \notin \mathcal{V}_1$ and $A \notin \mathcal{V}_2$, then its minimal polynomial needs to be $\lambda^2 - 2\lambda = \lambda(2 - \lambda)$, since none of its divisors cancels A.

(c) Find $\min\mathrm{P}_T$.

The least common multiple of $\lambda - 2$, λ and $\lambda(\lambda - 2)$ is $\min\mathrm{P}_T = \lambda(\lambda - 2)$.

(d) Show that T is diagonalizable.

This follows from the argument in (a)

4. (15 points) Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ be a bilinear map and let $v^* \in \mathcal{V}^*$.

(a) Show that $\psi(v_1, v_2) = (\Psi(v_1, v_2), v^*)$ defines a bilinear form.
For general $v_1, v_2, v_3 \in \mathcal{V}$ and $a, b \in \mathbb{F}$,

$$\begin{aligned}
\psi(av_1 + bv_2, v_3) =& (\Psi(av_1 + bv_2, v_3), v^*) \\
=& (a\Psi(v_1, v_3) + b\Psi(v_2, v_3), v^*) && (\Psi \text{ is bilinear}) \\
=& a(\Psi(v_1, v_3), v^*) + b(\Psi(v_2, v_3), v^*) && (v^* \text{ is linear}) \\
=& a\psi(v_1, v_3) + v\psi(v_2, v_3).
\end{aligned}$$

For identical reasons

$$\begin{aligned}
\psi(v_3, av_1 + bv_2) =& (\Psi(v_3, av_1 + bv_2), v^*) \\
=& (a\Psi(v_3, v_1) + b\Psi(v_3, v_2), v^*) && (\Psi \text{ is bilinear}) \\
=& a(\Psi(v_3, v_1), v^*) + b(\Psi(v_3, v_2), v^*) && (v^* \text{ is linear}) \\
=& a\psi(v_3, v_2) + v\psi(v_3, v_1),
\end{aligned}$$

for all $v_1, v_2, v_3 \in \mathcal{V}$ and $a, b \in \mathbb{F}$. This proves that ψ is bilinear.

(b) Apply this to show that $\psi(A, B) = \text{trace}(B^{Tr}A)$ is a bilinear form in $\mathcal{M}(n, \mathbb{R})$.
It is easy to prove that the map $\text{trace}(A) = a_{11} + \ldots + a_{nn}$ is a linear form in $\mathcal{M}(n, \mathbb{R})$. We next prove that $\Psi(A, B) = B^{Tr}A$ is bilinear. This follows from very simple properties of the product of matrices and by the linearity of the transposition operator: for all $A, B, C \in \mathcal{M}(n, \mathbb{R})$ and $a, b \in \mathbb{R}$

$$\begin{aligned}
\Psi(aA + bB, C) &= C^{Tr}(aA + bB) = aC^{Tr}A + bC^{Tr}B = a\Psi(A, C) + c\Psi(B, C), \\
\Psi(C, aA + bB) &= (aA + bB)^{Tr}C = aA^{Tr}C + bB^{Tr}C = a\Psi(C, A) + c\Psi(C, B).
\end{aligned}$$

Bilinearity of ψ follows from (a).

(c) Show that ψ is symmetric.
This is a simple argument:

$$\begin{aligned}
\psi(B, A) = \text{trace}(A^{Tr}B) &= \text{trace}((A^{Tr}B)^{Tr}) && (\text{since trace}\, C = \text{trace}\, C^{Tr}) \\
&= \text{trace}(B^{tr}A) = \psi(A, B) && ((CD)^{Tr} = D^{Tr}C^{Tr})
\end{aligned}$$

5. (15 points) Let $T \in \mathcal{L}(\mathcal{V})$ be represented by the matrix

$$A = \begin{bmatrix} c & 1 & & \\ & c & 1 & \\ & & c & 1 \\ & & & c \end{bmatrix}.$$

(The numbers that are not displayed are zeros.) Show that it can also be represented by the matrix

$$B = \begin{bmatrix} c & & & \\ 1 & c & & \\ & 1 & c & \\ & & 1 & c \end{bmatrix}.$$

In other words, show that A and B are similar.
If $\{v_1, v_2, v_3, v_4\}$ is the basis with respect to which the matrix representation is A, then

$$Tv_1 = cv_1, \qquad Tv_2 = cv_2 + v_1, \qquad Tv_3 = cv_3 + v_2, \qquad Tv_4 = cv_4 + v_3.$$

Let us then reorder the basis to get $\{v_4, v_3, v_2, v_1\}$ or, equivalently $w_j = v_{5-j}$ for $j = 1, \ldots, 4$. Then

$$Tw_1 = cw_1 + w_2, \qquad Tw_2 = cw_2 + w_3, \qquad Tw_3 = cw_3 + w_4, \qquad Tw_4 = cw_4.$$

The matrix representation w.r.t. this basis is therefore B.

6. (15 points) An operator $T \in \mathcal{L}(\mathcal{V})$ is cyclic if there exists a basis $\{v_1, v_2, \ldots, v_n\}$ such that the associated matrix is

$$\begin{bmatrix} 0 & 0 & \ldots & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 0 & 1 & -a_{n-1} \end{bmatrix}.$$

In this case, the vector $v = v_1$ is called cyclic as well. Show that $\mathrm{minP}_{T,v} = (-1)^n \chi_T$. (Hint. Write what having this matrix representation means in terms of the vectors of the basis.) Use this to prove that T is cyclic if and only if $\chi_T = (-1)^n \mathrm{minP}_T$.

The characteristic polynomial of this matrix/operator is $P(x) = (-1)^n (x^n + \sum_j a_j x^j)$. It is simple to see from the matrix representation that

$$T^j v = v_{j+1} \qquad j = 0, \ldots, n-1,$$

so $\{v, Tv, \ldots, T^{n-1}v\} = \{v_1, \ldots, v_n\}$ are linearly independent. This proves that the degree of the minimal polynomial for v is n, and since it has to be monic and divide the characteristic polynomial it follows that $\mathrm{minP}_{T,v}(x) = x^n + \sum_j a_j x^j$.

If T is cyclic, then we have seen that there exists v such that $\mathrm{minP}_{T,v} = (-1)^n \chi_T$. Since minP_T is monic, divides χ_T and is a multiple of $\mathrm{minP}_{T,v}$ it follows that $\mathrm{minP}_T = \mathrm{minP}_{T,v}$.

Assume that $\chi_T = (-1)^n \mathrm{minP}_T$. We will see next that there exists v such that $\mathrm{minP}_{T,v} = \mathrm{minP}_T$ (this is true for any operator). If we consider the vectors $v_1 = v, v_2 = Tv, \ldots, v_n = T^{n-1}v$, it is clear that they are linearly independent (the degree of $\mathrm{minP}_{T,v}$ is n), so they form a basis for $\mathcal{V}$. Since $Tv_j = v_{j+1}$ for $j = 1, \ldots, n-1$, the matrix representation of T w.r.t. this basis has the given form.

Side problem: there exists v such that $\mathrm{minP}_{T,v} = \mathrm{minP}_T$. Let

$$\mathrm{minP}_T = \Phi_1^{m_1} \ldots \Phi_k^{m_k},$$

where Φ_j are relatively prime and irreducible (they cannot be factored). For each j, there exists v_j such that $\mathrm{minP}_{T,v_j} = \Phi_j^{m_j} Q_j$ for some polynomial Q_j. Otherwise, the factor $\Phi_j^{m_j}$ would not be in the minimal polynomial. Define then $w_j = Q_j(T)v_j$. It is very simple to see that $\mathrm{minP}_{T,w_j} = \Phi_j^{m_j}$. Finally, it is relatively easy to prove that $v = w_1 + \ldots + w_k$ has minP_T as its minimal polynomial.