NAME:

MATH 672: Vector spaces

Fall'13 | Quiz # 8 | Partial solutions

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1. (30 points) Let $T \in \mathcal{L}(\mathcal{V})$, $\{v_1, \ldots, v_n\}$ be a basis for $\mathcal{V}$ and assume that

$$Tv_j = v_1 + v_2 + \ldots + v_n \qquad \forall j.$$

(a) Show that $v_1 + \ldots + v_n$ is an eigenvector.
This is quite obvious by linearity

$$T(v_1 + v_2 + \ldots + v_n) = n(v_1 + v_2 + \ldots + v_n)$$

(b) Compute $\nu(T)$.
The matrix associated to T in basis $\{v_1, \ldots, v_n\}$ is

$$A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

Its rank is clearly one, so $\nu(T) = n - 1$.

(c) Using the results of (a) and (b) (no computations are required), give χ_T.
From (a), $\lambda = n$ is an eigenvalue, and from (b) we have $n - 1$ linearly independent eigenvectors for $\lambda = 0$. Therefore

$$\chi_T(\lambda) = (-1)^n(\lambda - n)\lambda^{n-1}.$$

(d) Show that T is diagonalizable.
This follows directly from (a) and (b), since we can find a basis composed of n linearly independent eigenvectors. Actually, it is easy to show that

$$\{v_1 + \ldots + v_n, v_2 - v_1, \ldots, v_n - v_1\}$$

is one such basis.

(e) Compute $\min\mathrm{P}_{T,v_j}$ for all j. (Hint. This is very easy using the definition.)
We have

$$T^2 v_j = T(v_1 + \ldots + v_n) = n(v_1 + \ldots + v_n) = nTv_j,$$

so $\min\mathrm{P}_{T,v_j}(\lambda)$ is a multiple of $\lambda(\lambda - n)$. However, since v_j is not en eigenvector, it cannot be any proper divisor, and therefore $\min\mathrm{P}_{T,v_j}(\lambda) = \lambda(\lambda - n)$.

(f) Write (and argument) what $\min\mathrm{P}_T$ is.
Since all basis vectors have the same minimal polynomial, it follows that $\min\mathrm{P}_T = \min\mathrm{P}_{T,v_j}$ for all j.

2. (15 points) Let A be such that

$$A^2 = 4A - 4I.$$

(a) What is the characteristic polynomial of A? (Prove your assertion.)
We have that $P(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$ satisfies $P(A) = 0$. Therefore the only possible root of χ_A is $\lambda = 2$ and $\chi_A(\lambda) = (2 - \lambda)^n$

(b) What are the possible minimal polynomials of A?
It can only be $\lambda - 2$ and $(\lambda - 2)^2$.

(c) Show that either A is diagonal or it is not diagonalizable.
If $\mathrm{minP}_A(\lambda) = \lambda - 2$, then $A = 2I$. Otherwise, A cannot be diagonalizable, since its only possible eigenvalue is 2 and in case of being diagonalizable it would be similar to $2I$ which is only similar to itself.

3. (10 points) Let $T \in \mathcal{L}(\mathcal{V})$ be given by its action on a basis

$$Tv_1 = 2v_1, \qquad Tv_2 = \alpha v_1 + 2v_2, \qquad Tv_3 = \beta v_1 + \gamma v_2 + 5v_3.$$

Show that T is diagonalizable if and only if $\alpha = 0$. (Hint. There is no need to compute eigenvectors. You just need to count how many linearly independent eigenvectors there are.)
The associated matrix is

$$A = \begin{bmatrix} 2 & \alpha & \beta \\ 0 & 2 & \gamma \\ 0 & 0 & 5 \end{bmatrix},$$

and therefore $\chi_T(\lambda) = (2 - \lambda)^2(5 - \lambda)$. It is clear that $\dim\ker(A - 5I) = 1$. Also

$$\rho(A - 2I) = \begin{cases} 2 & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0, \end{cases}$$

which proves that

$$\dim\ker(A - 2I) = \nu(T - 2I) = 3 - \rho(A - 2I) = \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 2 & \text{if } \alpha = 0, \end{cases}$$

and this finishes the proof.

4. (10 points) In $\mathbb{R}_3[x]$ we consider the inner product

$$\langle p, q \rangle = \int_0^1 x p(x) q(x) dx.$$

Compute a basis for $\{1, x\}^\perp$.
This is equivalent to finding all polynomials $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that

$$\tfrac{1}{2}a_0 + \tfrac{1}{3}a_1 + \tfrac{1}{4}a_2 + \tfrac{1}{5}a_3 = 0, \qquad \tfrac{1}{3}a_0 + \tfrac{1}{4}a_2 + \tfrac{1}{5}a_2 + \tfrac{1}{6}a_3 = 0.$$

The solutions to this system are

$$a_2 \begin{bmatrix} \frac{3}{10} \\ -\frac{6}{5} \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} \frac{2}{5} \\ -\frac{6}{5} \\ 0 \\ 1 \end{bmatrix}, \qquad a_2, a_3 \in \mathbb{R},$$

so $\{\frac{3}{10} - \frac{6}{5}x + x^2, \frac{2}{5} - \frac{6}{5}x + x^3\}$ is a basis for $\{1, x\}^\perp$.

5. (15 points) Let $\mathcal{H}$ be a complex inner product space endowed with the product $\langle \cdot, \cdot \rangle$. Let $\{v_1, \dots, v_k\}$ be linearly independent vectors and consider the $k \times k$ matrix A with elements

$$a_{ij} = \langle v_i, v_j \rangle.$$

(This is called a Gram matrix.)

(a) Show that A is Hermitian. (This is defined as $\overline{A}^{Tr} = A$.)
Since $\overline{a}_{ji} = \overline{\langle v_j, v_i \rangle} = \langle v_i, v_j \rangle = a_{ij}$, the result is straightforward.

(b) If $c \in \mathbb{C}^k_c$, find $v \in \mathcal{H}$ such that

$$\|v\|^2 = \overline{c}^{Tr} Ac.$$

Show that $\overline{c}^{Tr} Ac \geq 0$ and $\overline{c}^{Tr} Ac = 0$ if and only if $c = 0$.
Using sesquilinearity, we show that

$$\overline{c}^{Tr} Ac = \sum_{ij} \overline{c}_i \langle v_i, v_j \rangle c_j = \langle \sum_i \overline{c}_i v_i, \sum_j \overline{c}_j v_j \rangle = \|v\|^2, \qquad v = \sum_j \overline{c}_j v_j.$$

Therefore $\overline{c}^{Tr} Ac \geq 0$ for all c. If $\overline{c}^{Tr} Ac = 0$, then $v = \sum_j \overline{c}_j v_j = 0$, but given that the vectors v_j are linearly independent (they are orthonormal), it follows that $c_j = 0$ for all j.

(c) By looking at the matrix A, how can you know if the set $\{v_1, \dots, v_k\}$ is orthogonal/orthonormal?
The set is orthogonal if and only if A is diagonal. It is orthonormal if and only if $A = I$.

6. (10 points) **The parallelogram law.** Show that if $\mathcal{H}$ is an inner product space with inner product $\langle \cdot, \cdot \rangle$, then its associated norm $\| \cdot \|$ satisfies:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

This is a very easy proof using (sesqui)linearity:

$$\begin{aligned} \|u + v\|^2 + \|u - v\|^2 =& \langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\ =& \|u\|^2 + \langle v, u \rangle + \langle u, v \rangle + \|v\|^2 \\ &+ \|u\|^2 - \langle v, u \rangle - \langle u, v \rangle + \|v\|^2 \\ =& 2\|u\|^2 + 2\|v\|^2. \end{aligned}$$

7. (10 points) **The real polarization formula.** Let $\mathcal{H}$ be a real vector space endowed with a norm $\| \cdot \|$ satisfying the parallelogram law (in addition to the axioms that define a norm):

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Show that

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)$$

is an inner product in $\mathcal{H}$ whose associated norm is $\| \cdot \|$. (You have to prove that the bracket $\langle \cdot, \cdot \rangle$ satisfies all the axioms that define a real inner product.)
We first note that (using the property $\|w\| = \| - w\|$)

$$\langle v, u \rangle = \tfrac{1}{4}(\|v + u\|^2 - \|v - u\|^2) = \tfrac{1}{4}(\|u + v\|^2 - \|u - v\|^2) = \langle u, v \rangle,$$

that is we have symmetry. Also

$$\langle u, u\rangle = \tfrac{1}{4}\|2u\|^2 = \|u\|^2,$$

so $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$. (This follows from the axioms defining a norm.)

Linearity in the first variable (the only one we need to check) is quite tricky. We first prove the following:

$$\begin{aligned}
\langle u_1, v_1\rangle + \langle u_2, v_2\rangle =& \tfrac{1}{4}\big(\|u_1+v_1\|^2 + \|u_2+v_2\|^2 - \|u_1-v_1\|^2 - \|u_2-v_2\|^2\big) \quad \text{(definition)}\\
=& \tfrac{1}{8}\big(\|u_1+u_2+v_1+v_2\|^2 + \|u_1-u_2+v_1-v_2\|^2 \\
& \quad - \|u_1+u_2-v_1-v_2\|^2 - \|u_1-u_2-v_1+v_2\|^2\big)^2 \qquad \text{(parallelogram law)}\\
=& \tfrac{1}{2}\big(\langle u_1+u_2, v_1+v_2\rangle + \langle u_1-u_2, v_1-v_2\rangle\big).
\end{aligned}$$

For easy reference, we repeat the formula:

$$\langle u_1, v_1\rangle + \langle u_2, v_2\rangle = \tfrac{1}{2}\big(\langle u_1+u_2, v_1+v_2\rangle + \langle u_1-u_2, v_1-v_2\rangle\big). \tag{1}$$

Using (1) we obtain

$$\langle u, v\rangle = \langle u, v\rangle + \langle u, 0\rangle = \tfrac{1}{2}\langle 2u, v\rangle$$

(since $\langle u, 0\rangle = 0$ as follows from the definition of the bracket), and by symmetry

$$\langle 2u, v\rangle = 2\langle u, v\rangle = \langle u, 2v\rangle. \tag{2}$$

Therefore

$$\begin{aligned}
\langle u_1, v\rangle + \langle u_2, v\rangle =& \tfrac{1}{2}\langle u_1+u_2, 2v\rangle \qquad \text{(by (1) and } \langle w, 0\rangle = 0)\\
=& \langle u_1+u_2, v\rangle, \qquad \text{(by (2))}
\end{aligned}$$

which proves additivity in the first variable. Using this property $n-1$ times, we show that

$$\langle n\, u, v\rangle = \langle u + \ldots + u, v\rangle = n\langle u, v\rangle \qquad \forall n \in \mathbb{Z}, \quad n \geq 0,$$

and also

$$\langle u, v\rangle + \langle -u, v\rangle = 0$$

so

$$\langle nu, v\rangle = n\langle u, v\rangle \qquad \forall n \in \mathbb{Z}. \tag{3}$$

Applying this to $w = nu$, we obtain

$$\tfrac{1}{n}\langle w, v\rangle = \langle \tfrac{1}{n}w, v\rangle \qquad \forall 0 \neq n \in \mathbb{Z}. \tag{4}$$

As a simple consequence of (3) and (4), we can show

$$\langle qu, v\rangle = q\langle u, v\rangle \qquad \forall q \in \mathbb{Q}. \tag{5}$$

The extension of (5) to real scalars need a continuity argument. Note first that

$$\begin{aligned}
\langle u, v\rangle =& \tfrac{1}{2}(\|u+v\|^2 - \|u\|^2 - \|v\|^2) \qquad \text{(parallelogram law)}\\
\leq& \tfrac{1}{2}\big((\|u\| + \|v\|)^2 - \|u\|^2 - \|v\|^2\big) \qquad \text{(triangle inequality)}\\
=& \|u\|\|v\|,
\end{aligned}$$

which (after applying this inequality to $-u$ and v, noticing that $\|-u\| = \|u\|$), yileds the Cauchy-Schwarz inequality

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Finally, let $r \in \mathbb{R}$ and $q \in \mathbb{Q}$. Then

$$\begin{aligned}
|\langle ru, v\rangle - r\langle u, v\rangle| &= |\langle ru, v\rangle - \langle qu, v\rangle - (r-q)\langle u, v\rangle| && \text{(by (5))} \\
&= |\langle (r-q)u, v\rangle - (r-q)\langle u, v\rangle| && \text{(additivity)} \\
&\leq |\langle (r-q)u, v\rangle| + |r-q||\langle u, v\rangle| \\
&\leq \|(r-q)u\|\|v\| + |r-q|\|u\|\|v\| && \text{(Cauchy-Schwarz)} \\
&= 2|r-q|\|u\|\|v\|.
\end{aligned}$$

Since we can take a sequence of rational numbers $q_n \in \mathbb{Q}$ such that $\lim_{n\to\infty} |r - q_n| = 0$, ti follows that

$$\langle ru, v\rangle = r\langle u, v\rangle.$$