I teach myself ... Hilbert spaces

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This document will be growing with the semester. Every in red is for you to justify. Even if we start with the basic definition of inner product, it is assumed that you know: what orthogonal means; what the Gram-Schmidt orthogonalization method is. We also assume that you know what convergence in a normed space means and any basic tool on normed spaces.

Week # 1

1 Definitions

Let H be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product is a map $(\cdot,\cdot) : H \times H \to \mathbb{K}$ (where $\mathbb{K}$ is the field) safisfying the following properties.

(a) Conjugate symmetry: $(x, y) = \overline{(y, x)}$ for all $x, y \in H$;

(b) Linearity in the first component: $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{K}$;

(c) Positivity: $(x, x) \geq 0$ for all $x \in H$ and $(x, x) = 0$ if and only if $x = 0$.

Note that in the complex case (a) implies that (x, x) is a real number, and therefore the hypothesis (c) makes sense. A map $(\cdot,\cdot)$ satisfying the properties (a)-(c) above is conjugate linear in its second component: $(x, \alpha y + \beta z) = \overline{\alpha}(x, y) + \overline{\beta}(x, z)$ for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{K}$. In the real case, we can define an inner product as a symmetric positive definite bilinear form. In the complex case, we say Hermitian positive definite sesquilinear (since the map is conjugate linear (anti-linear) in the second component) form. To avoid having to say everything twice (for the real and complex case) we will use sesquilinear and Hermitian in both cases. By (c), (x, x) is a non-negative real number and we can define

$$\|x\| := (x, x)^{1/2}.$$

Proposition 1.1 (Cauchy–Schwarz inequality). *A Hermitian positive definite sesquilinear form satisfies*

$$|(x, y)| \leq \|x\| \, \|y\| \qquad \forall x, y \in H.$$

Proof. Assume that $\|y\| = 1$. Then, for all $\lambda \in \mathbb{K}$,

$$0 \leq (x + \lambda y, x + \lambda y) = \|x\|^2 + |\lambda|^2 + 2\mathrm{Re}\left(\overline{\lambda}\,(x, y)\right).$$

Taking $\lambda = -(x, y)$, it follows that

$$|(x, y)| \leq \|x\| \quad \forall x \in H \quad \text{if } \|y\| = 1.$$

For general y we distinguish two cases: if $y = 0$ there is nothing to prove; if $y \neq 0$ we use the previous inequality applied to x and $\frac{1}{\|y\|}y$. $\square$

The Cauchy-Schwarz inequality is instrumental in proving that $\|\cdot\|$ is a norm in H. In fact, the only non-trivial part of the proof of the axioms for a norm is the triangle inequality, which follows from the Cauchy-Schwarz inequality. A **Hilbert space** is an inner product space that is complete as a normed space.
Note that in an inner product space, because of the Cauchy-Schwarz inequality

$$|(x_n, y) - (x, y)| \leq \|x_n - x\|\,\|y\|,$$

and we have that $x_n \to x$ implies $(x_n, y) \to (x, y)$ for all y. In other words, if $\lim_{n\to\infty} x_n$ exists, then

$$\lim_{n\to\infty}(x_n, y) = (\lim_{n\to\infty} x_n, y) \quad \forall y.$$

2 Orthonormal sequences and Bessel's inequality

Consider (assuming it exists) a countable sequence $(\phi_n)_{n\geq 1}$ in an inner product space H satisfying

$$(\phi_n, \phi_m) = \delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

We then say that this sequence (or set) is orthonormal.
In any infinite-dimensional inner product space, we can find orthonormal sequences. This can be done by using the Gram-Schmidt orthogonalization method (which we assume to be known to the reader) applied to a linearly independent sequence of elements of the space. For $x \in H$ we define

$$P_N x = \sum_{n=1}^{N} (x, \phi_n)\phi_n \in \mathrm{span}\{\phi_1, \ldots, \phi_N\}.$$

It is simple to see that $P_N x$ is orthogonal to $x - P_N x$ and therefore

$$\|x\|^2 = \|P_N x\|^2 + \|x - P_N x\|^2.$$

Additionally

$$\|P_N x\|^2 = \sum_{n=1}^{N} |(x, \phi_n)|^2 \leq \|x\|^2.$$

Therefore, we have the inequality (**Bessel's inequality**)

$$\sum_{n=1}^{\infty} |(x, \phi_n)|^2 \le \|x\|^2 \qquad \forall x \in H.$$

As a consequence of Bessel's inequality, it is easy to prove that the sequence $P_N x$ is a Cauchy sequence in H. If we assume that H is a Hilbert space, the series

$$Px := \sum_{n=1}^{\infty} (x, \phi_n)\phi_n = \lim_{N\to\infty} P_N x = \lim_{N\to\infty} \sum_{n=1}^{N} (x, \phi_n)\phi_n \tag{2.1}$$

converges in H. Additionally, for all $x \in H$,

$$(Px, \phi_n) = (x, \phi_n) \quad \forall n, \qquad \text{and} \qquad \|Px\|^2 = \sum_{n=1}^{\infty} |(x, \phi_n)|^2.$$

Two remarks. As a consequence of Bessel's inequality, if $(\phi_n)_{n\ge 1}$ is an orthonormal sequence, then

$$\lim_{n\to\infty} (x, \phi_n) = 0 \qquad \forall x \in H.$$

An interesting fact of orthonormal sequences is based on the fact that

$$\|\phi_n - \phi_m\| = \sqrt{2} \qquad n \ne m.$$

Therefore, an orthonormal sequence does not contain Cauchy (and therefore convergent) subsequences. In other words, if $(\phi_n)_n$ is an orthonormal sequence, the set $\{\phi_n : n \ge 1\}$ is not relatively compact. This gives a very simple proof of the fact that the unit ball in an infinitely dimensional Hilbert space is not relatively compact.

3 A prototypical example

Consider the set ℓ^2 of all sequences of complex numbers $\mathbf{c} = (c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

Since

$$|a_n + b_n|^2 \le 2|a_n|^2 + 2|b_n|^2,$$

it is easy to show that ℓ^2 is a vector space. Also, since

$$|a_n b_n| \le \tfrac{1}{2}|a_n|^2 + \tfrac{1}{2}|b_n|^2,$$

the inner product of sequences

$$(\mathbf{a}, \mathbf{b})_2 := \sum_{n=1}^{\infty} a_n \overline{b_n},$$

is well defined. The associated norm is

$$\|\mathbf{a}\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}.$$

Proposition 3.1. *ℓ^2 is a Hilbert space*

Proof. Let $(\mathbf{a}^k)_{k\ge 1}$ be a sequence of elements of ℓ^2. (In this proof, we will use a super-index to count the sequence, and a sub-index to count the elements of each of the $\mathbf{a}^k = (a_n^k)_{n=1}^{\infty}$. If $(\mathbf{a}^k)_k$ is Cauchy in ℓ^2, then, since

$$|a_n^k - a_n^l| \le \|\mathbf{a}^k - \mathbf{a}^l\|_2 \qquad \forall n$$

it is clear that the limits

$$a_n^{\infty} := \lim_{k\to\infty} a_n^k$$

exist for all n. It is also clear that for all $\varepsilon > 0$ there exists k_0 such that

$$\sum_{n=1}^{N} |a_n^k - a_n^l|^2 \le \varepsilon \qquad \forall k, l \ge k_0, \qquad \forall N.$$

Taking the limit as $l \to \infty$, while keeping k fixed,

$$\sum_{n=1}^{N} |a_n^k - a_n^{\infty}|^2 \le \varepsilon \qquad \forall k \ge k_0, \qquad \forall N.$$

Therefore,

$$\lim_{k\to\infty} \|\mathbf{a}^k - \mathbf{a}^{\infty}\|_2 = 0.$$

In a final step, we derive that $\mathbf{a}^{\infty} \in \ell^2$ and show that $\lim_{k\to\infty} \mathbf{a}^k = \mathbf{a}^{\infty}$. □

An interesting orthonormal set in ℓ^2 is composed by the canonical sequences $\mathbf{e}_n = (\delta_{m,n})_{m=1}^{\infty}$. Note that

$$(\mathbf{a}, \mathbf{e}_n) = a_n \qquad \forall \mathbf{a} \in \ell^2, \quad \forall n.$$

Therefore, $(\mathbf{a}, \mathbf{e}_n) = 0$ for all n implies $\mathbf{a} = 0$.

Week # 2

4 Countable Hilbert bases

Let H be a Hilbert space and $(\phi_n)_{n\ge 1}$ be an orthonormal sequence. We say that this set is a complete orthonormal sequence in H when (in additional to being orthonormal)

$$(x, \phi_n) = 0 \quad \forall n \qquad \Longrightarrow \qquad x = 0.$$

If we define Px as in Section 2

$$Px = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n,$$

then, using the fact that $(x - Px, \phi_n) = 0$ for all n, it is clear that

$$x = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n \qquad \forall x \in H. \tag{4.1}$$

Bessel's inequality, when applied to a complete orthonormal sequence becomes **Parseval's identity**

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, \phi_n)|^2 \qquad \forall x \in H.$$

This is just a reflection of the series decomposition of x, after computing the norm of the right had side of (4.1). Using the series for two elements of H, we can prove that

$$(x, y) = \sum_{n=1}^{\infty} (x, \phi_n)(\phi_n, y) \qquad \forall x, y \in H.$$

We can now give a more abstract point of view of what happens when we have a complete orthonormal set (also known as a **countable Hilbert basis**) in a Hilbert space H. We have a map

$$H \ni x \longmapsto Tx := \big((x, \phi_n)\big)_{n=1}^{\infty} \in \ell^2.$$

This map is defined if $(\phi_n)_{n\geq 1}$ is an orthonormal sequence. It is quite clear that this is a linear operator. When the sequence is complete (and only when it is complete, since this is the definition), the operator T is injective. It is actually an isometry

$$\|x\| = \|Tx\|_2,$$

by Parseval's identity. It is quite easy to show that T is surjective, by proving that if $\mathbf{c} = (c_n)_{n=1}^{\infty} \in \ell^2$, the series

$$\sum_{n=1}^{\infty} c_n \phi_n$$

converges (we just need to show that it is Cauchy) and, if we call it sum x, then $c_n = (x, \phi_n)$ for all n. We thus have an isometric isomorphism between H and ℓ^2, satisfying also

$$(Tx, Ty)_2 = (x, y) \qquad \forall x, y \in H.$$

We can go a little further and state the following result:

Theorem 4.1. *Let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in a Hilbert space H. The following statements are equivalent:*

(a) *$(\phi_n)_{n\geq 1}$ is complete orthonormal.*

(b) $x = \sum_{n=1}^{\infty}(x, \phi_n)\phi_n$ *for all* $x \in H$.

(c) $\|x\|^2 = \sum_{n=1}^{\infty} |(x, \phi_n)|^2$ *for all* $x \in H$.

Proof. We have already seen that (a) implies (b) by showing that $x - \sum_{n=1}^{\infty}(x, \phi_n)\phi_n$ is orthogonal to all the ϕ_n. We have also seen that (b) implies (c), since the norm of the partial sums converges to the norm of the series. Finally, if (c) holds, it is clear that $(x, \phi_n) = 0$ for all n, implies $x = 0$. $\square$

5 Separability

A legitimate question that arises from the previous section is whether countable Hilbert bases ever exist, or, even better, what kind of Hilbert spaces have countable Hilbert bases. We already know that ℓ^2 has a countable Hilbert basis and that any Hilbert space admitting a countable Hilbert space is isometrically isomorphic to ℓ^2. We will see that these spaces can be characterized by a topological property. We say that H is **separable**, when H has a countable set that is dense in H, that is, when there exists a countable subset $V = \{v_n : n \geq 1\} \subset H$ (a subset, not a subspace) such that for all $x \in H$ and $\varepsilon > 0$, there exists n such that $\|x - v_n\| < \varepsilon$. In other words, for every x there exists a sequence of elements extracted from V and converging to x. Note that we do not use the word subsequence, because all the elements of V are different, while we can have sequences with repeated elements.

Proposition 5.1. *If a Hilbert space H admits a countable Hilbert basis, then it is separable.*

Proof. Let $x \in H$ and $\varepsilon > 0$. We can then find N such that

$$\|x - \sum_{n=1}^{N}(x, \phi_n)\phi_n\| \leq \varepsilon.$$

For every $n \in \{1, \dots, N\}$, we can find $q_n \in \mathbb{Q} + \imath\mathbb{Q}$ such that

$$|(x, \phi_n) - q_n| \leq \frac{\varepsilon}{2^n}.$$

Therefore

$$\|x - \sum_{n=1}^{N} q_n\phi_n\| < \varepsilon + \varepsilon\left(\sum_{n=1}^{N}\frac{1}{4^n}\right)^{1/2} \leq \left(1 + \frac{1}{\sqrt{3}}\right)\varepsilon.$$

The previous computation shows that the set

$$M = \{\sum_{n=1}^{N} q_n\phi_n \;:\; N \geq 1, \quad q_n \in \mathbb{Q} + \imath\mathbb{Q}\}$$

is dense in H. However, this set is countable. $\square$

Proposition 5.2. *If H is a separable Hilbert space, then H admits a countable Hilbert basis.*

Proof. Let $V = \{v_n : n \geq 1\}$ be a countable dense set. Let us apply the Gram-Schmidt orthogonalization process to the sequence $(v_n)_{n\geq 1}$. This sequence does not need to have linearly independent elements, so it can happen that some steps of the Gram-Schmidt process do not produce an element of the orthonormal set. In principle, it might happen that what we get is a finite orthonormal set (ϕ_n). (We will see this cannot happen if H is infinite-dimensional at the very end.) Let now $x \in H$ satisfy

$$(x, \phi_n) = 0 \quad \forall n.$$

Then

$$(x, v_n) = 0 \quad \forall n,$$

since the elements v_n are linear combinations of the elements of the orthonormal set (ϕ_n). We can write this last condition as

$$(x, v) = 0 \quad \forall v \in V.$$

Since V is dense in H, there exists a sequence $(w_n)_{n\geq 1}$ of elements in V such that $x = \lim_{n=1}^{\infty} w_n$. Then

$$0 = \lim_{n\to\infty} (x, w_n) = (x, \lim_{n\to\infty} w_n) = \|x\|^2,$$

and we have proved that (ϕ_n) is complete orthonormal. If H is infinite-dimensional, (ϕ_n) cannot be finite because that would mean that every x can be represented as a linear combination of the elements of (ϕ_n). □

We can round up this section with another result that relates separability of a Hilbert space with its being isomorphic to ℓ^2.

Proposition 5.3. *Let H be a Hilbert space. If H is isomorphic to ℓ^2, then H is separable.*

Proof. The hypothesis imply that there exists a bounded bijection $R : \ell^2 \to H$. Note that, since R is bounded, there exists $C > 0$ such that

$$\|R\mathbf{c}\| \leq C\|\mathbf{c}\|_2 \quad \forall {\in} \ell^2.$$

We know that ℓ^2 has a dense countable subset V. This implies that $R(V) = \{Rv : v \in V\}$ is also countable and dense in H. □

In summary, for a Hilbert space H, we have proved that the following statements are equivalent:

(a) H is separable,

(b) H is isomorphic to ℓ^2,

(c) H has a countable Hilbert basis.

Week #3

6 The orthogonal projection

Let H be a Hilbert space and let M be a **closed subspace** of H. Let $x \in H$ and consider the quantity

$$\delta := \inf_{y \in M} \|x - y\| \geq 0,$$

which can be considered as the distance of x to M. By definition of δ, there exists a sequence of elements of M, $(y_n)_{n\geq 1}$, such that

$$\|x - y_n\| \to \delta.$$

By the parallelogram identity,

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|(x - y_n) - (x - y_m)\|^2 \\
&= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\left\|x - \tfrac{1}{2}(y_n + y_m)\right\|^2 \\
&\leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\delta^2.
\end{aligned}$$

It is easy to prove from the above inequality that $(y_n)_{n\geq 1}$ is a Cauchy sequence. Therefore, there exists $y \in M$ such that $y = \lim_{n\to\infty} y_n$ (since H is Hilbert and M is closed). In particular,

$$y \in M \qquad \text{satisfies} \qquad \|x - y\| \leq \|x - z\| \qquad \forall z \in M.$$

We can continue the computation before writing a definition and a corresponding theorem. Let $z \in M$ be such that $\|z\| = 1$. Then

$$\begin{aligned}
\|x - y\|^2 &\leq \|x - (y + (x - y, z)z)\|^2 \\
&= \|x - y\|^2 + |(x - y, z)|^2 - 2|(x - y, z)|^2,
\end{aligned}$$

which, after elementary simplifications implies that $(x - y, z) = 0$. We have then shown that if $y \in M$ satisfies $\|x - y\| \leq \|x - z\|$ for all $z \in M$, then

$$(x - y, z) = 0 \qquad \forall z \in M.$$

Proposition 6.1. *Let M be a closed subspace of a Hilbert space H. Given $x \in H$, the problems:*

$$\text{find} \quad y \in M \quad \text{such that} \quad \|x - y\| \leq \|x - z\| \quad \forall z \in M, \tag{6.1}$$

and

$$\text{find} \quad y \in M \quad \text{such that} \quad (x - y, z) = 0 \quad \forall z \in M \tag{6.2}$$

are equivalent and uniquely solvable.

Proof. We have already shown that problem (6.1) admits a solution and that its solution is a solution of (6.2). If $y \in M$ is a solution of (6.2), then

$$\|x - z\|^2 = \|x - y + (y - z)\|^2 = \|x - y\|^2 + \|y - z\|^2 \geq \|x - y\|^2 \qquad \forall z \in M,$$

and y solves (6.1). We have then shown that both problems are equivalent and that (6.1) has a solution. Let us finally show that (6.1) admits only one solution. Let $y, y' \in M$ solve (6.2). Since $y' - y \in M$, we have

$$(x - y, y' - y) = 0 \qquad (x - y', y' - y) = 0.$$

Subtracting these equalities it follows that $\|y' - y\|^2 = 0$ and therefore the solution is unique. $\square$

The operator such $P : H \to H$ given by $x \mapsto y = Px$, where y the solution of (6.2), is called the **orthogonal projection onto** M**.** This operator finds the best approximation of x in the closed subspace M. Because of the characterization (6.2), it is clear that P is a linear operator. Note that we can write (6.2) in the slightly different form

$$\text{find} \quad Px \in M \quad \text{such that} \quad (Px, z) = (x, z) \quad \forall z \in M.$$

An exact computation. We had already met the orthogonal projection in a very particular instance: if $(\phi_n)_{n\geq 1}$ is an orthonormal sequence in H, and M is the closure of $\text{span}\{\phi_n \,:\, n \geq\}$, then

$$Px = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n \in M$$

satisfies $(x - Px, \phi_n) = 0$ for all n. By linearity

$$(x - Px, z) = 0 \qquad \forall z \in \text{span}\{\phi_n \,:\, n \geq 1\}.$$

Finally, takin a limit it is easiy to show that

$$(x - Px, z) = 0 \qquad \forall z \in M = \overline{\text{span}\{\phi_n \,:\, n \geq 1\}}.$$

We have thus shown that Px is the orthogonal projection onto the closure of the span of the orthonormal sequence $(\phi_n)_{n\geq 1}$. Reciprocally, if M is a closed subspace of H admitting a countable Hilbert basis $(\phi_n)_{n\geq 1}$, then

$$Px = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n$$

is the orthogonal projection of x on M.
We end this section with some simple properties of the orthogonal projection.

Proposition 6.2. *Let $P : H \to H$ be the orthogonal projection onto a closed subspace M of a Hilbert space H. Then*

(a) $Px = x$ *if and only if* $x \in M$.

(b) $P^2 = P$.

(c) $\|Px\| \leq \|x\|$ *for all* $x \in H$.

(d) $(Px, y) = (Px, Py) = (x, Py)$ *for all* $x, y \in H$.

Proof. Property (a) is a simple consequence of the fact that (6.2) defines the orthogonal projection, and (b) follows from (a). To show (c), just note that

$$\|x\|^2 = \|x - Px + Px\|^2 = \|x - Px\|^2 + \|Px\|^2.$$

The proof of (d) is simple. Since

$$(Px, y) = \overline{(y, Px)} = \overline{(Py, Px)} = (Px, Py),$$

we can introduce the operator P acting on y. The rest of the proof follows from the definition of Px. $\square$

Week # 4

7 Orthogonal decompositions

In this section we will study simultaneous orthogonality to sets of vectors and subspaces. Let us start with a Hilbert space H and a subset $A \subset H$ (A does not need to be a subspace). We define

$$A^\perp := \{x \in H \,:\, (x, a) = 0 \quad \forall a \in A\}.$$

It is very easy to see that $A^\perp$ is a subspace of H, even if A is not. Moreover, if we wave a sequence $(x_n)_{n\geq 1}$ in $A^\perp$, since

$$|(x, a)| = |(x - x_n, a)| \leq \|x_n - x\| \, \|a\| \quad \forall a \in A,$$

it follows that $A^\perp$ is closed. The argument can be mimicked in the second component (using convergent sequences in the second component), to show that

$$(x, a) = 0 \quad \forall a \in A \qquad \Longleftrightarrow \qquad (x, a) = 0 \quad \forall a \in \overline{A},$$

and therefore $A^\perp = (\overline{A})^\perp$.
We can now consider a **closed subspace** $M \subset H$. If $P : H \to H$ is the orthogonal projection onto M (Section 6), then

$$x = Px + (x - Px), \tag{7.1}$$

where $Px \in M$ and $x - Px \in M^\perp$. Moreover, $M \cap M^\perp = \{0\}$. This gives a direct orthogonal decomposition

$$H = M \oplus M^\perp,$$

which is attained using the orthogonal projection (7.1). We emphasize that for this to be true M has to be closed.
If M is a **general subspace** of H, then $\overline{M}$ is a closed subspace and we can decompose, orthogonally,

$$H = \overline{M} \oplus (\overline{M})^\perp = \overline{M} \oplus M^\perp.$$

There is yet another reading of this decomposition, which requires a little more thinking. Given x, we have already seen that $x - Px \in M^\perp$, where $Px \in \overline{M}$ is the orthogonal projection of x onto $\overline{M}$. Moreover

$$(x - (x - Px), z) = (Px, z) = 0 \qquad \forall z \in M^\perp,$$

which, by Proposition 6.1, shows that $x - Px$ is the projection of x onto the closed subspace $M^\perp$. Therefore, the two orthogonal decompositions

$$H = \overline{M} \oplus M^\perp = M^\perp \oplus M^{\perp\perp}, \qquad M^{\perp\perp} := \{x \in H \,:\, (x, y) = 0 \quad \forall y \in M^\perp\}$$

which shows the equality of the spaces

$$M^{\perp\perp} = \overline{M}.$$

Week # 5

8 The Riesz-Fréchet representation theorem

Let H be a Hilbert space. A very unconventional form of writing the Cauchy-Schwarz inequality is the following

$$\|x\| = \sup_{0 \neq y \in H} \frac{|(x, y)|}{\|y\|} \qquad \forall x \in H.$$

Note that the Cauchy-Schwarz inequality implies that

$$\sup_{0 \neq y \in H} \frac{|(x, y)|}{\|y\|} \leq \|x\|,$$

but that taking $x = y$ we get equality. Moreover, this is one of the few happy instances of the supremum being a maximum. Consider now the space

$$H^* := \{\ell : H \to \mathbb{K} \,:\, \ell \text{ linear and bounded}\} = \mathcal{B}(H; \mathbb{K}),$$

which endowed with the norm

$$\|\ell\|_{H^*} := \sup_{0 \neq x \in H} \frac{|\ell(x)|}{\|x\|}$$

is a Banach space. (This is known from the theory of linear operators between normed spaces.) Consider now the following map:

$$\begin{array}{rcl} \jmath : H & \longrightarrow & H^* \\ x & \longmapsto & (\,\cdot\,, x), \end{array}$$

where, for fixed $x \in H$, $(\,\cdot\,, x)$ denotes the map

$$H \ni y \longmapsto (y, x) \in \mathbb{K}.$$

Given what we have written before, it is not difficult to show that $\jmath : H \to H^*$ is

(a) well defined,

(b) conjugate linear,

(c) injective,

(d) isometric.

The Riesz-Fréchet theorem asserts that j is actually an isometric isomorphism of H and H^*. This will have some additional consequences.

Theorem 8.1 (Riesz-Fréchet)**.** *The map j is surjective. In other words, for all $f \in H^*$ there exists a unique $x \in H$ such that*

$$(y, x) = f(y) \quad \forall y \in H.$$

Moreover, $\|x\| = \|f\|_{H^*}$ and the inverse map $j^{-1} : H^* \to H$ is conjugate linear and isometric.

Proof. If $f \neq 0$, then $\mathrm{Ker} f$ is a closed proper subspace of H and, therefore there exists $0 \neq x_0 \in (\mathrm{Ker} f)^\perp$. Note that, since $f(x_0) \neq 0$, we can decompose

$$y = \underbrace{\left(y - \frac{f(y)}{f(x_0)} x_0\right)}_{\in \mathrm{Ker}\, f} + \underbrace{\frac{f(y)}{f(x_0)} x_0}_{\in \mathrm{span}\{x_0\}} \qquad \forall y \in H,$$

with orthogonal sum. Since

$$(y, x_0) = \frac{f(y)}{f(x_0)} \|x_0\|^2 \qquad \forall y \in H,$$

it follows that

$$x = \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0$$

satisfies $(y, x) = f(y)$ for all y. Finally, the inverse of $\jmath$ is conjugate linear since

$$\jmath^{-1}(\alpha f + \beta g) = \jmath^{-1}\left(\jmath(\overline{\alpha}\jmath^{-1}(f) + \overline{\beta}\jmath^{-1}(g))\right) = \overline{\alpha}\jmath^{-1}(f) + \overline{\beta}\jmath^{-1}(g).$$

□

Corollary 8.2. *Every Hilbert space is reflexive.*

Week # 9

Using the Riesz-Fréchet Theorem and the parallelogram identity, we can easily prove the following result:

Corollary 8.3. *The dual of a Hilbert space is a Hilbert space.*

9 The Hilbert space adjoint

The Riesz-Fréchet theorem provides a very simple form of dealing with adjoints of bounded linear operators. As we will see, this approach will produce an adjoint operator that is different (but strongly related) to the adjoint operator obtained with dual spaces.
If we have two Hilbert space X and Y and a bounded linear operator $\Lambda : X \to Y$, then the expression

$$(\Lambda x, y)_Y = (x, \Lambda^* y)_X \qquad \forall x \in X \quad \forall y \in Y \tag{9.1}$$

can be used to formally define an operator $\Lambda^* : Y \to X$ which is called the Hilbert space adjoint of Λ. Let us see how. Fix $y \in Y$ and consider the map

$$X \ni x \longmapsto \phi_y(x) := (\Lambda x, y)_Y.$$

Note that $\phi_y \in X^*$ and

$$\|\phi_y\|_{X^*} \leq \|\Lambda\| \|y\|_Y.$$

Therefore, by the Riesz-Fréchet representation theorem, there exists and element of x, that we denote $\Lambda^* y$, such that

$$(x, \Lambda^* y)_X = \phi_y(x) = (\Lambda x, y)_Y \qquad \forall x \in X.$$

Moreover $\|\Lambda^* y\|_Y \leq \|\Lambda\| \|y\|_Y$. It is easy to show (the Riesz-Fréchet theorem says there is a unique element...) that Λ^* is linear.) We have thus prove the existence of a bounded operator $\Lambda^* : Y \to X$ satisfying (9.1). We have also proved that $\|\Lambda^*\| \leq \|\Lambda\|$. A simple uniqueness argument shows that $\Lambda^{**} := (\Lambda^*)^* = \Lambda$ and therefore

$$\|\Lambda\| = \|\Lambda^*\|.$$

Relation to the Banach space adjoint. Let the angled bracket $\langle \phi, x \rangle_{X^* \times X}$ represent the action $\phi(x)$ for any $\phi \in X^*$, and similarly for Y^* and Y. Let $\jmath_X : X \to X^*$ and $\jmath_Y : Y \to Y^*$ be the Riesz-Fréchet maps associated to the spaces X and Y. (The Riesz-Fréchet representation map is the inverse map.) These maps are conjugate linear isometric bijections. It can then be seen that if $\Lambda^* : Y \to X$ is the Hilbert space adjoint of Λ, then

$$\jmath_x \circ \Lambda^* \circ \jmath_Y^{-1} : Y^* \to X^*$$

is linear, bounded (with the same norm as $\|\Lambda)$ and satisfies

$$\langle (\jmath_x \circ \Lambda^* \circ \jmath_Y^{-1})(y^*), x \rangle_{X^* \times X} = \langle y^*, \Lambda x \rangle_{Y^* \times Y} \qquad \forall x \in X, y^* \in Y^*$$

From operators to sesquilinear forms and back. Let now $A : X \to Y$ be bounded and $A^* : Y \to X$ be its adjoint. The following map $a : X \times Y \to \mathbb{K}$

$$a(x, y) := (Ax, y)_Y = (x, A^* y)_X$$

is sesquilinear and it satisfies

$$|a(x, y)| \le C\, \|x\|_X \|y\|_Y \qquad \forall x \in X, \quad y \in Y \tag{9.2}$$

with $C = \|A\|$. We say that a sesquilinear map satisfying (9.2) is bounded. Let now $a : X \times Y \to \mathbb{K}$ be a bounded sesquilinear map. When we freeze the first variable we obtain a bounded linear operator

$$\mathcal{A}x := \overline{a(x, \cdot)} : Y \to \mathbb{K}$$

and we can therefore find an element $Ax \in Y$ such that

$$\overline{(Ax, \cdot)_Y} = (\cdot, Ax)_Y = \mathcal{A}x = \overline{a(x, \cdot)}.$$

In other words, there exists a unique $Ax \in Y$ such that

$$(Ax, y)_Y = a(x, y) \qquad \forall x \in X, \quad \forall y \in Y. \tag{9.3}$$

Moreover,

$$\|Ax\|_Y = \|\mathcal{A}x\|_{Y^*} \le C\|x\|_X. \tag{9.4}$$

Using a simple uniqueness argument, it follows that the map $A : X \to Y$ given by (9.3) is linear and bounded by (9.4). The smallest constant $C > 0$ such that (9.2) (and therefore (9.4)) holds is the norm of the operator A. If we repeat the argument fresszing the second variable, we recover the adjoint operator A^*.

10 Weak convergence

In a normed space X, weak convergence (denoted $x_n \rightharpoonup x$) is seen through the dual space

$$\phi(x_n) \longrightarrow \phi(x) \qquad \forall \phi \in X^*.$$

Thanks to the Riesz-Fréchet theorem, in a Hilbert space H weak convergence can be seen through the inner product

$$(x_n, y) \longrightarrow (x, y) \quad \forall y \in H.$$

Note that even if we already know that strong convergence implies weak convergence, in Hilbert space we have an alternative proof using the Cauchy-Schwarz inequality

$$|(x_n, y) - (x, y)| = |(x_n - x, y)| \le \|x_n - x\|\, \|y\|.$$

An example. We also have a very simple counter-example of a weakly convergence sequence that does not converge strongly: if $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in H, then $\phi_n \rightharpoonup 0$ but $\|\phi_n\| = 1$ and therefore ϕ_n does not converge to 0. (To see the weak convergence to zero, think of the Bessel inequality.) Note that $\|\phi_n - \phi_m\| = \sqrt{2}$ if $n \neq m$, and therefore $(\phi_n)_{n\geq 1}$ cannot contain convergent subsequences either. This fact can also be proved by noticing that any convergent subsequence will necessarily converge to zero (because the whole sequence converges wealy to zero), but the elements have unit norm.

Weak convergence and bounded operators. Let $\Lambda : X \to Y$ be bounded. We know that a convergence sequence $x_n \to x$ is mapped to a convergent sequence $\Lambda x_n \to \Lambda x$. Since $(\Lambda x_n, y) = (x_n, \Lambda^* y)$, it is also clear that $x_n \rightharpoonup x$ implies $\Lambda x_n \rightharpoonup \Lambda x$.

Weak convergence and boundedness. Invoking the Banach-Steinhaus Theorem (Uniform Boundedness Principle) we know that every weakly convergent sequence is bounded. By the Banach-Alaoglu Theorem, it follows that every bounded sequence has a weakly convergent subsequence. This might not be entirely obvious at first sight, given that the Banach-Alaoglu Theorem deals with bounded sequences in the dual space and $*-$weak convergence. Here is a simple argument to make both ends meet. Let $(x_n)_{n\geq 1}$ be a bounded sequence in a Hilbert space H and let $\jmath : H \to H^*$ be the Riesz-Fréchet map. Applying the Banach-Alaoglu Theorem to $\jmath x_n$, we can find a weakly convergent subsequence $x_{n_k} \rightharpoonup x$.

Theorem 10.1. *Weakly convergent sequences in a Hilbert space are bounded. Reciprocally, every bounded sequence in a Hilbert space contains a weakly convergent subsequence.*