Your first day at work – MATH 806 (Fall 2015)

1. Let X be a set (with no particular algebraic structure). A function $d : X \times X \to \mathbb{R}$ is called a metric on X (and then X is called a metric space) when d satisfies the following axioms:

A sequence $(x_n) \subset X$ is said to converge to $x \in X$ when

Sequences in a metric space can only have one limit. The proof is based on this simple inequality

$$d(x, x') \le d(x, x_n) + d(x', x_n).$$

Finish it.

A sequence (x_n) is said to be a Cauchy sequence (we typically just say the the sequence is Cauchy) when

Every convergent sequence is Cauchy. The proof is based on this simple inequality

$$d(x_n, x_m) \le d(x_n, x) + d(x_m, x).$$

Finish it.

When in a metric space X, every Cauchy sequence is convergent, we say that X is complete.

2. Let now V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is said to be a norm when it satisfies the following axioms:

When a vector space is equipped with a norm, it is said to be a normed space. Given a norm on a vector space V, we can easily define a metric/distance on V:

The metric brings along the concepts of convergent and Cauchy sequence. A normed space that is complete is called a **Banach space.** Not every normed space is a Banach space.

Remark. Not every metric space is a normed space. To begin with, a metric space does not neet to have a vector structure, while a normed space does. Even on vector spaces, a metric derived from a norm takes values in the entire $[0, \infty)$, while many metrics take values in bounded intervals.

3. Consider now a vector space V and a function $(\cdot, \cdot) : V \times V \to \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, depending on the field over which V is defined. The bracket is called an inner product when it satisfies the following axioms:

Given an inner product, we can define the associated norm as follows:

Given an inner product and its associated norm, the following inequality, known as the Cauchy-Schwarz inequality, holds:

The proof in the real case is very simple. Fix $u, v \in V$ and consider the map

$$\mathbb{R} \ni t \longmapsto (u + t\,v, u + t\,v) \in \mathbb{R}.$$

It is easy to show that this map is a quadratic polynomial with at most one zero. The CS inequality follows from this simple argument. The proof for the complex case is slightly more involved. Using the Cauchy-Schwarz inequality it is easy to prove that every inner product space is a normed space, i.e., the associated norm is actually a norm. We therefore have concepts of convergent and Cauchy sequences. An inner product space that is complete is called a **Hilbert space**.

Remark. Not every inner product space is a normed space. The norm associated to an inner product satisfies the paralelogram identity:

but not every norm satisfies this identity. Not every inner product space is a Hilbert space.

Quiz # 1 — September 9 – MATH 806 (Fall 2015)

Your name:

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1. (2 points) Define Banach space. (No formulas are needed for this definition.)

2. (2 points) Let H be an inner product space and let $\|x\| = (x, x)^{1/2}$ be the associated norm. State the Cauchy-Schwarz inequality.

3. (3 points) Let X be a metric space. Give two equivalent definitions of what we understand by a compact subset of X.

(Wait to be told before you start with the next page.)

4. (3 points) Let (X, d) be a metric space and let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Assume that there exists a convergent subsequence $(x_{n_k})_{k\geq 1}$ (here $(n_k)_{k\geq 1}$ is an increasing sequence of positive integers). Show that $(x_n)_{n\geq 1}$ converges.

(You can discuss Problem 4 with one classmate.
You are not allowed to go back to Questions 1–3)

Quiz # 2 — September 16 – MATH 806 (Fall 2015)

Your name:

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1. (1 point) Define Hilbert space. (No formulas are needed for this definition.)

2. (1 point) Let H be an inner product space and $\|x\| = (x,x)^{1/2}$, where $(\cdot,\cdot)$ is the inner product in H. Write the Cauchy-Schwarz inequality.

3. (2 points) Let $(x_n)_{n\geq 1}$ be a convergent sequence in an inner product space H. If $x = \lim_{n\to\infty} x_n$, show that

$$(x_n, y) \longrightarrow (x, y) \quad \forall y \in H.$$

4. (3 points) Let H be a Hilbert space and $M \subset H$ be a subspace. Show that the set

$$M^\perp := \{x \in H \,:\, (x,m) = 0 \quad \forall m \in M\}$$

is a **closed subspace** of H.

5. (3 points) Define the spaces ℓ^p for $1 \leq p < \infty$ and for $p = \infty$. Define their norms.

6. (2 points) Write down Minkowski's inequality for sequences.

7. (3 points) Show that $\ell^p \subset \ell^\infty$ for every $p \in [1, \infty)$. Find $\mathbf{x} \in \ell^\infty$ such that $\mathbf{x} \notin \ell^p$ for any p.

8. (5 points) Let $K \in L^2(\Omega \times \Omega)$ and consider the operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ given by

$$(\Lambda f)(x) = \int_\Omega K(x, y) f(y) dy.$$

(Here Ω is an open set in $\mathbb{R}^d$.) Show that Λ is bounded and

$$\|\Lambda\| \leq \|K\|_{L^2(\Omega\times\Omega)}.$$

Quiz # 3 — September 23 – MATH 806 (Fall 2015)

Your name:

1. (3 points) Let $A : X \to Y$ be a bounded linear operator between two normed spaces X and Y. Give three different but equivalent definitions of the operator norm $\|A\|$.

2. (2 points) Let $\|\cdot\|_*$ and $\|\cdot\|_\circ$ be two norms defined in a vector space X. What do we mean when we say that these norms are equivalent?

3. (2 points) The Cauchy-Schwarz inequality states that in an inner product space H,

$$|(x, y)| \le \|x\|\,\|y\| \qquad \forall x, y \in H,$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product and $\|\cdot\|$ the associated norm. Give necessary and sufficient conditions on x and y so that the inequality is an equality.

4. (2 points) Show that in an inner product space H

$$\|x\| = \sup_{0 \neq y \in H} \frac{|(x,y)|}{\|y\|} \qquad \forall x \in H.$$

Is the supremum a maximum?

5. (2 points) Find an element of ℓ^2 that is not in ℓ^1.

6. (2 points) Let $g \in L^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^d$. Consider the operator

$$L^p(\Omega) \ni f \longmapsto \Lambda f := g\, f \in L^p(\Omega),$$

where $p \in [1, \infty)$. Show that Λ is bounded and $\|\Lambda\| \leq \|g\|_{L^\infty(\Omega)}$. (Note that they are actually equal, but I am not asking you for the proof of equality.)

7. Let H be an infinite dimensional inner product space and let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in H. Such a sequence can always be built using the Gram-Schmidt method applied to a sequence of linearly independent elements of H. (Do no prove this!)

   (a) (2 points) Show that $(\phi_n)_{n\geq 1}$ does not contain Cauchy subsequences.

   (b) (1 points) Use (a) to give a direct proof that in an infinite dimensional inner product space the unit ball is not compact.

8. (3 points) Let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in a Hilbert space H. Bessel's inequality states that

$$\sum_{n=1}^{\infty} |(x, \phi_n)|^2 \leq \|x\|^2 \qquad \forall x \in H.$$

Use this inequality to show that the series

$$\sum_{n=1}^{\infty} (x, \phi_n)\phi_n$$

converges in H for all $x \in H$. (Hint. Show that the sequence of partial sums is Cauchy.)

Quiz # 4 — September 30 – MATH 806 (Fall 2015)

Your name:

1. (2 points) Let X be a normed space. Define its dual X^* and the norm in X^*.

2. (3 points) Let $A : X \to Y$ and $B : Y \to Z$ be bounded operators. Show that $B\,A : X \to Z$ is bounded and

$$\|B\,A\| \leq \|B\|\,\|A\|.$$

3. (3 points) Let X be a complex normed space and let V be a subspace of X. What does the extension theorem say? (Note that the full statement says two things about the outcome.)

4. (3 points) Let M be a non-empty subset of an inner product space H. Show that

$$M^{\perp} := \{x \in H \,:\, (x, m) = 0 \quad \forall m \in M\}$$

is a closed subspace of H.

5. (3 points) Let V be a subspace of a normed space X. Prove that if 0 is an interior point to V, then $V = X$. (Hint. Build a ball around the origin.)

6. (6 points) Let H be a Hilbert space and $(\phi_n)_{n\geq 1}$ be an orthonormal sequence. Let $(\alpha_n)_{n\geq 1} \in \ell^2$.

(a) Show that the elements

$$s_N := \sum_{n=1}^{N} \alpha_n \phi_n$$

define a Cauchy sequence in H. (Hint. Compute $\|s_N - s_M\|^2$ when $M > N$.)

(b) Compute

$$\|s_N\|^2 \qquad \text{and} \qquad \lim_{N\to\infty} \|s_N\|.$$

(c) Use (a) and (b) to prove that the map

$$\ell^2 \ni \alpha = (\alpha_n)_{n\geq 1} \longmapsto T\alpha := \sum_{n=1}^{\infty} \alpha_n \phi_n \in H$$

is an isometry from ℓ^2 to H.

The Baire Category Theorem

Let X be a complete metric space. We say that and $V \subset X$ is dense in X when $\overline{V} = X$ or, equivalently, when $V \cap \Omega \neq \emptyset$ for all non-empty open $\Omega \subset X$. Consider now a sequence of open dense subsets $(V_k)_{k\geq 1}$ of X. Take Ω non-empty open and $x_0 \in \Omega$. There exists $r_0 > 0$ (which we can assume to satisfy $r_0 \leq 1$) such that

$$B(x_0, 3r_0) := \{x \in X \,:\, d(x, x_0) < 3r_0\} \subset \Omega.$$

For every $k \geq 1$, we can find $x_k \in X$ and $r_k > 0$ such that

$$B(x_k, 3r_k) \subset V_k \cap B(x_{k-1}, r_{k-1}).$$

Prove it.

By construction we can make

$$r_k \leq \frac{r_{k-1}}{3} \leq \ldots \leq \frac{r_0}{3^k} \leq \frac{1}{3^k}.$$

Prove it.

Note that $x_k \in B(x_{k-1}, r_{k-1})$ for all k and therefore $d(x_{k+1}, x_k) \leq 1/3^k$. With this it is simple to show that the sequence $(x_k)_{k\geq 1}$ is Cauchy. Prove it.

Let x^* be the limit of this sequence (X is complete). Then

$$d(x^*, x_k) \leq \sum_{j=k}^{\infty} d(x_{j+1}, x_j) \qquad \forall k.$$

Prove it.

This implies that

$$d(x^*, x_k) \leq \frac{3}{2} r_k \qquad \forall k,$$

and therefore $x^* \in B(x_k, 3r_k) \subset V_k$. Prove it.

Note that we have show that there exists a point x^* such that

$$x^* \in \cap_{k=1}^{\infty} V_k.$$

However, $d(x^*, x_0) \leq \frac{3}{2} r_0$ and therefore $x_0 \in B(x_0, 3r_0) \subset \Omega$. Summing up, for every open set Ω, there exists

$$x^* \in \Omega \cap (\cap_{k=1}^{\infty} V_k).$$

This proves that $\cap_{k=1}^{\infty} V_k$ is dense in X. This completes the proof of the following result:

> **Baire Category Theorem.** *The intersection of a sequence of dense open sets in a complete metric space is dense in the space. In other words, if X is a complete metric space and V_k are open dense subsets of X for all $k \geq 1$, then $\cap_{k=1}^{\infty} V_k$ is dense in X.*

Let now F_n be closed subsets of a complete metric space. Prove that if the interior of F_n is empty for all n, that is, $\mathring{F}_n = \emptyset$, then $\cup_{n=1}^{\infty} F_n \neq X$. (Hint. Use the Baire category theorem with $V_n := X \setminus F_n$. You only need to use that $\cap_{n=1}^{\infty} V_n$ is not empty.)

Rephrased in a slightly different (while equivalent) way, we have proved the following well-known consequence of the Baire Category Theorem.

> **Proposition.** *If X is a complete metric space and $X = \cup_{n=1}^{\infty} F_n$ with F_n closed for all n, then there exists n such that $\mathring{F}_n \neq \emptyset$.*

The Banach-Steinhaus Theorem

Today X and Y will be two generic Banach spaces. Let now $\Lambda \in \mathcal{B}(X;Y)$. Prove the inequality

$$\|\Lambda x\|_Y \geq \|\Lambda x_0\|_Y - \|\Lambda\| \, \|x - x_0\|_X \qquad \forall x, x_0 \in X$$

and use it to show that for all $C > 0$ the set

$$S_C := \{x \in X \, : \, \|\Lambda x\|_Y > C\}$$

is open.

Show that if there exists $x_0 \in X$ and $\rho > 0$ such that

$$S_C \cap B(x_0;\rho) = \emptyset, \qquad \text{where } B(x_0;\rho) := \{x \in X \, : \, \|x - x_0\|_X < \rho\}$$

then, using the inequality

$$\|\Lambda x\|_Y \leq \|\Lambda x_0\|_Y + \|\Lambda(x + x_0)\|_Y \qquad \|x\|_X \leq \rho,$$

we can prove that

$$\|\Lambda\| \leq \frac{2C}{\rho}.$$

Let now $\mathcal{F} \subset \mathcal{B}(X;Y)$ be an arbitrary collection of bounded linear operators from X to Y. Show that

$$V_n := \{x \, : \, \|\Lambda x\|_Y > n \quad \text{for some } \Lambda \in \mathcal{F}\}$$

is open.

Show that if V_n is not dense in X, then

$$\sup_{\Lambda \in \mathcal{F}} \|\Lambda\| < \infty.$$

Use the previous arguments and Baire's Theorem[1] to prove the following result:

Banach-Steinhaus Theorem (a.k.a. Uniform Boundedness Principle)
If X and Y are Banach spaces and $\mathcal{F} \subset \mathcal{B}(X;Y)$, then either

$$\sup_{\Lambda \in \mathcal{F}} \|\Lambda\| < \infty$$

or the set

$$\{x \in X \,:\, \sup_{\Lambda \in \mathcal{F}} \|\Lambda x\|_Y = \infty\}$$

is dense in X.

Note that the following result is a partial statement of the Banach-Steinhaus Theorem

Let $\mathcal{F} \subset \mathcal{B}(X;Y)$, where X and Y are Banach spaces. If

$$\sup_{\Lambda \in \mathcal{F}} \|\Lambda x\|_Y =: C_x < \infty \qquad \forall x \in X,$$

then the set $\mathcal{F}$ is bounded in the operator norm.

[1] The intersection of a sequence of dense open sets in a complete metric space is dense in the space. In other words, if X is a complete metric space and V_n are open dense subsets of X for all $n \geq 1$, then $\cap_{n=1}^{\infty} V_n$ is dense in X.

Prove the following corollary now (note that you need to prove linearity and boundedness):

Corollary. *Let* $(\Lambda_n)_{n\geq 1}$ *be a sequence in* $\mathcal{B}(X;Y)$, *where* X *and* Y *are Banach spaces. If the following limit exists*

$$\lim_{n\to\infty} \Lambda_n x =: \Lambda x \qquad \forall x \in X,$$

then $\Lambda : X \to Y$ *is linear and bounded.*

Note that the result does not make any claims on whether $\Lambda_n \to \Lambda$ (in the operator norm). The result can be read as: the pointwise limit of a sequence of bounded linear operator is a bounded linear operator.

To show the following result, identify $x \in X$ with $i\,x \in X^{**}$, where

$$(i\,x)(\phi) := \phi(x) \qquad x \in X, \quad \phi \in X^*, \qquad \|i\,x\|_{X^{**}} = \|x\|_X,$$

and use the previous corollary.

Corollary. *Let X be a Banach space and let $x_n \rightharpoonup x$. Then there exists $C > 0$ such that*

$$\|x_n\| \le C \qquad \forall n.$$

In other words, weakly convergent sequences are bounded.

Worksheet – October 28 – MATH 806 (Fall 2015)

1. Consider the operator $G : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ defined by

$$(Gf)(x) := \int_0^x f(t)\mathrm{d}t.$$

   (a) Show that $\|G\| = 1$.

   (b) Show that G is compact from $\mathcal{C}[0,1]$ to $\mathcal{C}[0,1]$. (Hint. Use the Arzelà-Ascoli theorem.)

   (c) Show that G does not have eigenvalues.

2. **Degenerate integral operators.** Let $a_j, b_j \in L^2(\Omega)$ for $j = 1, \ldots, N$ and let

$$K(x,y) := \sum_{j=1}^{N} a_j(x) b_j(y).$$

   Show that the operator

$$(Tf)(x) := \int_\Omega K(x,y) f(y)\mathrm{d}y$$

   is compact. Show that it can only have a finite number of non-zero eigenvalues.

3. Consider a kernel function $K \in \mathcal{C}(\overline{\Omega} \times \overline{\Omega})$, where Ω is a bounded set, and the associated integral operator

$$(Tf)(x) = \int_\Omega K(x,y) f(y)\mathrm{d}y.$$

   (a) Show that $T : L^1(\Omega) \to \mathcal{C}(\overline{\Omega})$ is bounded and give an estimate on $\|T\|$.

   (b) Show that T is compact as an operator defined between the same two spaces.

   (c) Give a one-line proof (based on (b)) that T defines a compact operator from $L^1(\Omega)$ to itself.

Quiz # 5 — November 4 – MATH 806 (Fall 2015)

Your name:

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1. (4 points) Show that the set of all linear compact operators between two Banach spaces X and Y is a closed subspace of $\mathcal{B}(X;Y)$.

2. (2 points) Give an example of a bounded non-compact linear operator. Justify your answer.

3. (4 points) Let $K : X \to X$ be a linear compact operator in a Banach space. Show that if $\lambda \neq 0$ is an eigenvalue of K, then the subspace of eigenvectors associated to λ is finite dimensional.

4. (4 points) Let X, Y be Banach spaces, $K : X \to Y$ be compact and $A : Y \to X$ be bounded. Show that AK and KA are compact.

5. (4 points) Consider the operator $K : \ell^1 \to \ell^1$ given by $(Kx_n) := a_n x_n$, where $a_n \to 0$. Show that K is compact.

6. (2 points) State Banach-Alaoglu's Theorem. Define everything that is needed to understand the statement of the theorem.

The Lax-Milgram Lemma

Let X be a real Hilbert space and $a : X \times X \to \mathbb{R}$ be a bilinear form satisfying

$$|a(u,v)| \leq M\|u\|\,\|v\| \qquad \forall u, v \in X.$$

(We say that a is a bounded bilinear form.) Then for every $u \in X$, the functional $a(u,\cdot)$ (given by $X \ni v \mapsto a(u,v) \in \mathbb{R}$) is linear bounded and satisfies

$$\|a(u,\cdot)\|_{X^*} \leq M\|u\|.$$

Therefore we can find $Au \in X$ such that

$$(Au, v) = a(u,v) \qquad \forall u, v \in X.$$

The map $A : X \to X$ thus defined is linear and bounded.

Let us now assume that the bilinear form satisfies the following property (coercivity): there exists $\alpha > 0$ such that

$$a(u, u) \geq \alpha \|u\|^2 \qquad \forall u \in X.$$

This inequality implies

$$\|Au\| \geq \alpha \|u\| \qquad \forall u \in X.$$

Then A is injective and has closed range.

We can also show that $(\text{Range}A)^{\perp} = \{0\}$.

The previous results prove the following:

> If $a : X \times X \to \mathbb{R}$ is a bounded coercive bilinear form in a real Hilbert space X, the operator $A : X \to X$ defined by
>
> $$(Au, v) = a(u, v) \qquad \forall u, v \in X$$
>
> is invertible. Furthermore, $\|A^{-1}\| \leq 1/\alpha$, where α is the coercivity constant of the bilinear form.

We can use this result to prove the Lax-Milgram lemma:

> Let X be a real Hilbert space and $a : X \times X \to \mathbb{R}$ be a bounded coercive bilinear form. Then for all $\ell \in X^*$, the problem
>
> $$\text{find } u \in X \quad \text{s.t.} \quad a(u, v) = \ell(v) \quad \forall v \in X$$
>
> has a unique solution and
>
> $$\|u\| \leq \|\ell\|_{X^*}/\alpha,$$
>
> where α is the coercivity constant.

(long) Quiz # 6 — November 11 – MATH 806 (Fall 2015)

Team members:

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1. (5 points) State the Banach-Steinhaus theorem.

2. (5 points) Let $(x_n)_{n\geq 1}$ be a weakly convergent sequence in a Hilbert space H. Show that it is bounded. (Hint. Consider the functionals $\phi_n := (\cdot, x_n) \in H^*$. Compute their norms and use the Banach-Steinhaus Theorem.)

3. (5 points) State the open mapping theorem.

4. (5 points) Show that in an infinite dimensional Banach space there are no open precompact sets (except the empty set).

5. (5 points) Show that a compact operator between infinite dimensional Banach spaces cannot be surjective.

6. (5 points) Let $\Lambda : X \to Y$ be a linear operator between two Banach spaces. Define the graph of Λ and explain what we mean when we say that the graph is closed. Finally, state the closed graph theorem.

7. (5 points) Define (do not derive its existence) the Hilbert space adjoint of a bounded linear operator $\Lambda : X \to Y$, where X and Y are Hilbert spaces.

8. (5 points) Let H be a complex Hilbert space, $I : H \to H$ be the identity map, and $\lambda \in \mathbb{C}$. What is the adjoint of $A = \lambda I$?

9. (5 points) Let X, Y be Hilbert spaces and let $\Lambda : X \to Y$ be a bounded linear operator. Show that

$$\operatorname{Ker} \Lambda = (\operatorname{Range} \Lambda^*)^{\perp}.$$

Here Λ^* is the Hilbert space adjoint and the symbol $\perp$ refers to orthogonality with respect to the inner product.

10. (5 points) Let $T : H \to H$ be a compact operator in a Hilbert space H. Show that $\operatorname{Ker}(T - I)$ is finite dimensional.