**MATH 806: Functional analysis**

Fall 2015 | Midterm exam (in-class part) | October 19

1. (10 points) Let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in an infinite dimensional Hilbert space.

   (a) State Bessel's inequality.

   **Solution.** If $H$ is a Hilbert space and $(\phi_n)_{n\geq 1}$ is an orthonormal sequence, then

   $$\sum_{n=1}^{\infty} |(x,\phi_n)|^2 \leq \|x\|^2 \qquad \forall x \in H,$$

   (where $(\cdot,\cdot)$ and $\|\cdot\|$ are the inner product in $H$ and the associated norm).

   (b) Define what we mean when we say that $(\phi_n)_{n\geq 1}$ is a complete orthonormal sequence.

   **Solution.** One possible definition: the set cannot be extended to a larger orthonormal set. Also

   $$(x,\phi_n) = 0 \quad \forall n \geq \qquad \Longrightarrow \qquad x = 0.$$

   (c) What happens to Bessel's inequality when $(\phi_n)_{n\geq 1}$ is a complete orthonormal sequence?

   **Solution.** It becomes an equality for every $x$. (That is another equivalent definition of complete orthonormal set.)

2. (10 points) Prove that in an inner product space $H$

   $$\|x\| = \sup_{0\neq y\in H} \frac{|(x,y)|}{\|y\|} \qquad \forall x \in H.$$

   **Solution.** The result needs to be proved for $x \neq 0$, since it is straightforward for $x = 0$. Then

   $$\|x\| = \frac{|(x,x)|}{\|x\|} \leq \sup_{0\neq y\in H} \frac{|(x,y)|}{\|y\|} \leq \sup_{0\neq y\in H} \frac{\|x\|\,\|y\|}{\|y\|} = \|x\|,$$

   by the Cauchy-Schwarz inequality.

3. (10 points) Let $(x_n)_{n\geq 1}$ be a sequence in a normed space $X$. Prove that if $x_n \to x$, then $x_n \rightharpoonup x$.

   **Solution.** If $x_n \to x$ and $\phi : X \to \mathbb{K}$ is continuous, then $\phi(x_n) \to \phi(x)$. Since this is true for all $\phi \in X^*$, it follows (by definition) that $x_n \rightharpoonup x$.

4. (10 points) Let $X$ be a Banach space and let the sequence $(x_n)_{n\geq 1}$ satisfy $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$. Show that $x = y$.

**Solution.** If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then

$$\phi(x_n) \to \phi(x) \qquad \text{and} \qquad \phi(x_n) \to \phi(y) \qquad \forall \phi \in X^*.$$

Since the limit is unique in $\mathbb{K}$, it follows that $\phi 9x) = \phi(y)$ for all $\phi \in X^*$. By the separation theorem, if $x \neq y$, then there exists $\phi \in X^*$ such that $\phi(x) \neq \phi(y)$. Therefore $x = y$ and the proof is finished.

5. (10 points) State Minkowski's inequality for sequences.

**Solution.** For every couple of sequences of complex numbers

$$\left(\sum_{n=1}^{\infty} |a_n + b_n|^p\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/2} + \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/2}.$$

6. (10 points) Consider the space of finite sequences

$$f := \{(x_n)_{n\geq 1} \,:\, x_n \in \mathbb{C}, \quad x_n = 0 \quad \forall n \geq N, \quad \text{for some } N\}.$$

Show that $f$ is dense in $\ell^p$ for all $p \in [1, \infty)$ but $f$ is not dense in $\ell^\infty$.

**Solution.** If $x = (x_n)_{n\geq 1} \in \ell^p$, with $1 \leq p < \infty$, then

$$\lim_N \sum_{n=N+1}^{\infty} |x_n|^p = 0.$$

Let now $y_N := (x_1, \ldots, x_N, 0, 0, \ldots) \in f$. Then

$$\|y_N - x\|_{\ell^p}^p = \sum_{n=N+1}^{\infty} |x_n|^p \to 0,$$

which proves that we can find a sequence of elements of $f$ converging to $x$ for every $x \in \ell^p$. Therefore $f$ is dense in $\ell^p$. When $p = \infty$, we can take $x = (1, 1, \ldots, 1, \ldots)$. If $y \in f$, then $y_n = 0$ for all $n > N$ for some $N$ and

$$\|y - x\|_{\ell^\infty} = \sup_n |y_n - x_n| \geq \sum_{n\geq N+1} |x_n| = 1.$$

This proves that $x$ cannot be approximated by any finite sequence.

7. (10 points) Starting on the inequality (do not prove it), valid for $q \geq p \geq 1$,

$$\left(\sum_{n=1}^{N} |x_n|^q\right)^{1/q} \leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \qquad \forall x_1, \ldots, x_N \in \mathbb{C}, \qquad \forall N,$$

show that the inclusion operator $i : \ell^p \to \ell^q$, given by $ix := x$, is bounded. What is the norm of this operator?

**Solution.** If $x = (x_n)_{n\geq 1} \in \ell^p$, then

$$\left(\sum_{n=1}^N |x_n|^q\right)^{1/q} \leq \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \qquad \forall N,$$

and therefore $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$. In particular, this proves that the operator $i$ is well defined, it is bounded (it is clearly linear) and satisfies $\|i\| \leq 1$. However, if $x = (1, 0, \ldots, 0, \ldots)$, then $\|x\|_{\ell^p} = 1 = \|x\|_{\ell^q}$, which proves that $\|i\| = 1$.

8. (10 points) Let $a := (a_n)_{n\geq 1} \in \ell^\infty$ and consider the operator $M_a : \ell^\infty \to \ell^\infty$ given by

$$(M_a x)_n := a_n\, x_n \quad \forall n \geq 1.$$

Show that $M_a \in \mathcal{B}(\ell^\infty; \ell^\infty)$ and $\|M_a\| = \|a\|_{\ell^\infty}$.

**Solution.** Linearity is simple to show: for any two sequences $x, y \in \ell^\infty$ and $\alpha, \beta \in \mathbb{K}$,

$$(M_a(\alpha x + \beta y))_n = a_n(\alpha x_n + \beta y_n) = \alpha a_n x_n + \beta a_n y_n = \alpha (M_a x)_n + \beta (M_a y)_n.$$

Also

$$|a_n x_n| \leq \|a\|_{\ell^\infty} \|x\|_{\ell^\infty} \qquad \forall n$$

and taking the supremum in the left-hand-side, it follows that $\|M_a x\|_{\ell^\infty} \leq \|a\|_{\ell^\infty}\|x\|_{\ell^\infty}$. This proves boundedness of $M_a$ and $\|M_a\| \leq \|a\|_{\ell^\infty}$. Finally, take $x = (1, 1, \ldots)$ and note that $M_a x = a$, while $\|x\|_{\ell^\infty} = 1$. Therefore

$$\|M_a\| \geq \|M_a x\|_{\ell^\infty} = \|a\|_{\ell^\infty}.$$

9. (10 points) Let $K \in L^\infty(\Omega \times \Omega)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^d$. Show that the operator

$$(Pf)(x) := \int_\Omega K(x, y)\, f(y)\, \mathrm{d}y$$

is bounded from $L^1(\Omega)$ to $L^p(\Omega)$ for all $p \in [1, \infty]$.

**Solution.** For almost every $x$ and $y$, $|K(x, y)| \leq M$, and therefore for almost every $x$:

$$|(Pf)(x)| \leq \int_\Omega |K(x, y)|\, |f(y)| \mathrm{d}y \leq M \int_\Omega |f(y)| \mathrm{d}y = M\|f\|_{L^1(\Omega)}.$$

If $p < \infty$, then

$$\int_\Omega |(Pf)(x)|^p \mathrm{d}x \leq M^p \|f\|^p_{L^1(\Omega)} |\Omega|,$$

where $|\Omega|$ is the measure of $\Omega$. Therefore

$$\|Pf\|_{L^p(\Omega)} \leq M|\Omega|^{1/p}\|f\|_{L^1(\Omega)} \qquad \forall f \in L^1(\Omega).$$

This proves that $P$ is bounded from $L^1(\Omega)$ to $L^p(\Omega)$ and $\|P\| \leq M|\Omega|^{1/p}$. (We can put $M = \|K\|_{L^\infty(\Omega\times\Omega)}$.) If $p = \infty$, the first inequality we have proved shows that

$$\|Pf\|_{L^\infty(\Omega)} \leq M\|f\|_{L^1(\Omega)} \qquad \forall f \in L^1(\Omega).$$

This proves that $P$ is bounded from $L^1(\Omega)$ to $L^\infty(\Omega)$ and $\|P\| \leq M$.

10. (10 points) Let $X$ be a normed space, $0 \neq x_0 \in X$, and $c \in \mathbb{R}^n$. Show that there exists $\Lambda : X \to \mathbb{R}^n$ linear and bounded satisfying $\Lambda x_0 = c$.

**Solution.** Using the separation theorem (or any version of the extension theorems), we can find $\phi \in X^*$ such that $\phi(x_0) \neq 0$. Therefore

$$\Lambda x := \frac{\phi(x)}{\phi(x_0)} c$$

satisfies the requirements. It is clearly linear and bounded, since

$$\frac{1}{\phi(x_0)} \phi \in X^*$$

and the map $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto t\, c$ is linear (and bounded). Finally $\Lambda x_0 = c$.

# MATH 806: Functional analysis

Fall 2015 Midterm exam (take home part) Due October 26

1. (5 points) Let $X$ and $Y$ be Banach spaces and let $(\Lambda_n)_{n\geq 1}$ be a sequence in $\mathcal{B}(X;Y)$. Show that if there exists $x$ such that $\|\Lambda_n x\| \to \infty$, then $\sup_n \|\Lambda_n\| = \infty$.

   **Solution.** If $\|\Lambda_n\| \leq C$ for all $n$, then $\|\Lambda_n x\| \leq C\|x\|$ for all $x$ and, therefore, $\|\Lambda_n x\|$ cannot diverge.

2. (5 points) Let $X$ and $Y$ be Banach spaces. Show that if $\Lambda \in \mathcal{B}(X;Y)$, then

$$x_n \rightharpoonup x \qquad \Longrightarrow \qquad \Lambda x_n \rightharpoonup \Lambda x.$$

   **Solution.** Given $\phi \in Y^*$, the map $\phi\Lambda : X \to \mathbb{K}$ is linear and bounded, that is, $\phi\Lambda \in X^*$. Therefore

$$\phi(\Lambda x_n) = (\phi\Lambda)(x_n) \to (\phi\Lambda)(x) = \phi(\Lambda x) \qquad \forall \phi \in Y^*,$$

   which means that $\Lambda x_n \rightharpoonup \Lambda x$.

3. (5 points) Let $X$ and $Y$ be normed spaces. Let $\Lambda : X \to Y$ be linear. Assume that we have defined boundedness as follows: $\Lambda$ is bounded when the image of $B(0;1) := \{x \in X : \|x\| < 1\}$ is bounded in $Y$. Show that $\Lambda$ is bounded if and only if the image by $\Lambda$ of any bounded set of $X$ is bounded in $Y$.

   **Solution.** One implication is straightforward, since $B(0;1)$ is bounded. Let $U \subset X$ be bounded. Then $U \subset B(0;R)$ for $R > 0$ large enough. Since

$$\|\Lambda x\| \leq C \qquad \forall x \in B(0;1),$$

   then

$$\|\Lambda(\tfrac{1}{R}z)\| \leq C \qquad \forall z \in B(0;R)$$

   and, by linearity

$$\|\Lambda z\| \leq RC \qquad \forall z \in B(0;R).$$

   This implies that $\Lambda(U) \subset B_Y(0; C\,R)$.

4. **The set of invertible operators is open.** (4 × 5 points) Let $X$ and $Y$ be Banach spaces.

   (a) *The Neumann series.* Show that if $\Lambda \in \mathcal{B}(X;X)$ satisfies $\|\Lambda\| < 1$, then the series

$$\sum_{n=0}^{\infty} \Lambda^n$$

   converges. (Here $\Lambda^0 = I$ is the identity operator.) Give an upper bound for the norm of the sum of the series.

**Solution.** Note that $\|\Lambda^n\| \leq \|\lambda\|^n$. (This can be easily proved by induction.) Therefore, for $N > M$,

$$\left\| \sum_{n=M+1}^{N} \Lambda^n \right\| \leq \sum_{n=M+1}^{N} \|\Lambda\|^n.$$

Since the following geometric series converges

$$\sum_{n=0}^{\infty} \|\Lambda\|^n = \frac{1}{1 - \|\Lambda\|},$$

given the fact that $\|\Lambda\| < 1$, then the estimate above shows that the sequence of partial sums $S_N := \sum_{n=0}^{N} \Lambda^n$ is Cauchy in $\mathcal{B}(X;X)$. However, $\mathcal{B}(X;X)$ is Banach (since $X$ is Banach) and this implies convergence of the series

$$S_N \to S := \sum_{n=0}^{\infty} \Lambda^n.$$

With the same arguments we can prove that

$$\|S_N\| = \left\| \sum_{n=0}^{N} \Lambda^n \right\| \leq \sum_{n=0}^{N} \|\Lambda\|^n \leq \sum_{n=0}^{\infty} \|\Lambda\|^n = \frac{1}{1 - \|\Lambda\|} \qquad \forall N,$$

and, since $S_N \to S$, then $\|S_N\| \to \|S\| \leq 1/(1 - \|\Lambda\|)$.

(b) Show that

$$(I - \Lambda)^{-1} = \sum_{n=0}^{\infty} \Lambda^n.$$

**Solution.** Let $S_N$ be as in (a). Then

$$(I - \Lambda)S_N = I - \Lambda^{N+1}.$$

Since $\|(I - \Lambda)(S_N - S)\| \leq \|I - \Lambda\| \|S_N - S\|$, it follows from the above that

$$(I - \Lambda)S_N \to (I - \Lambda)S \qquad (I - \Lambda)S_N = I - \Lambda^{N+1} \to I$$

(note that $\|\Lambda^{N+1}\| \leq \|\Lambda\|^{N+1}$), which proves that $(I - \Lambda)S = I$. A similar argument shows that $S(I - \Lambda) = I$.

(c) Let $A \in \mathcal{B}(X;Y)$ be invertible. Show that if $B \in \mathcal{B}(X;Y)$ satisfies

$$\|B\| < \frac{1}{\|A^{-1}\|},$$

then $A - B$ is invertible. (Hint. $A - B = A(I - A^{-1}B)$.)

**Solution.** By (b), using that $\|A^{-1}B\| \leq \|A^{-1}\| \, \|B\| < 1$, it follows that $I - A^{-1}B$ is invertible. Since $A$ is invertible, $A(I - A^{-1}B) = A - B$ is also invertible.

(d) Use (c) to show that the set

$$\{\Lambda \in \mathcal{B}(X;Y) \, : \, \Lambda \text{ is invertible}\}$$

is open in $\mathcal{B}(X;Y)$.

**Solution.** If $A$ is invertible, by (c), so is $A - B$ for every $B$ such that $\|B\| < 1/\|A^{-1}\|$. Therefore the ball $B(A; \frac{1}{\|A^{-1}\|})$ is contained in the set of invertible operators. This proves that the set of invertible operators is open.

5. **Extension of a densely defined operator.** ($5 \times 5$ points) Let $X$ and $Y$ be Banach spaces and let $\Lambda : D(\Lambda) \to Y$ be a linear operator defined on a subspace $D(\Lambda) \subset X$ satisfying:

$$\overline{D(\Lambda)} = X \qquad \|\Lambda x\|_Y \le C\|x\|_X \qquad \forall x \in D(\Lambda).$$

(a) Given $x \in X$ and a sequence $(x_n)_{n\ge1}$ in $D(\Lambda)$ such that $x = \lim_n x_n$, show that the limit

$$\lim_n \Lambda x_n$$

exists.

**Solution.** From the inequality

$$\|\Lambda(x_n - x_m)\|_Y \le C\|x_n - x_m\|_X,$$

it is clear that the sequence $(\Lambda x_n)_{n\ge1}$ is Cauchy, and therefore convergent, in $Y$.

(b) Show that if $\lim_n x_n = x = \lim_n x'_n$, where $(x_n)_{n\ge1}$ and $(x'_n)_{n\ge1}$ are sequences in $D(\Lambda)$, then

$$\lim_n \Lambda x_n = \lim_n \Lambda x'_n.$$

(Hint. Consider the sequence $(x_1, x'_2, x_3, x'_4, ....)$.)

**Solution.** For all $\varepsilon > 0$, there exists $N$ such that

$$\|x - x_n\| < \varepsilon \qquad \|x - x'_n\| < \varepsilon \quad \forall n \ge N.$$

Ths proves that the combined sequence

$$\widetilde{x}_n := \begin{cases} x_n & \text{if } n \text{ is odd,} \\ x'_n & \text{if } n \text{ is even,} \end{cases}$$

is convergent to $x$. Therefore $\lim \Lambda\widetilde{x}_n$ exists. However, this sequence contains the subsequences $(\Lambda x_{2n-1})_{n\ge1}$ and $(\Lambda x'_{2n})_{n\ge1}$, so both converge to the same limit. Since these subsequences are also respective subsequences of the convergent sequences $(\Lambda x_n)_{n\ge1}$ and $(\Lambda x'_n)_{n\ge1}$, the limits of the latter have to coincide.

We then define $Ax := \lim_n \Lambda x_n$.

(c) Show that $Ax = \Lambda x$ for $x \in D(\Lambda)$.

**Solution.** If $x \in D(\Lambda)$, we can take the sequence with elements $x_n := x$ for all $n$. Therefore $Ax = \lim \Lambda x_n = \lim \Lambda x = \Lambda x$.

(d) Show that

$$\|Ax\|_Y \le C\|x\|_X \quad \forall x \in X.$$

**Solution.** If $(x_n)_{n\geq 1}$ is a sequence of elements of $D(\Lambda)$ converging to $x \in X$, then $\|x_n\|_X \to \|x\|_X$ and $\|\Lambda x_n\|_Y \to \|Ax\|_Y$. Taking the limit in the inequality

$$\|Ax_n\|_Y \leq C\|x_n\|_X \quad \forall n,$$

the result follows.

(e) Show that if $B \in \mathcal{B}(X;Y)$ satisfies

$$Bx = \Lambda x \quad \forall x \in D(\Lambda),$$

then $B = A$.

**Solution.** Let $(x_n)_{n\geq 1}$ be a sequence in $D(\Lambda)$ converging to $x$. Then, since $A$ and $B$ are linear and bounded $Ax = \lim Ax_n$ and $Bx = \lim Bx_n$. However, since $x_n \in D(\Lambda)$ and $A$ and $B$ extend $\Lambda$, then $Ax_n = \Lambda x_n = Bx_n$ and the result is proved.

6. **The space $\mathcal{C}^\infty(\mathbb{R})$.** (20 points) Consider the space $X := \mathcal{C}^\infty(\mathbb{R})$ of infinitely differentiable functions of a real variable. Define a metric in $X$ with the following property: $f_n \to f$ if and only if

$$f_n^{(j)} \to f^{(j)} \quad \text{uniformly on compact sets of } \mathbb{R}, \qquad \text{for all } j \geq 0.$$

(Prove that the metric you define has actually that property.) Show that $X$ is a Fréchet space when endowed with such a metric. (Note that there are many metrics providing the same concept of convergence in $X$.)

**Solution.** [Sketch only]
(a) *Construction.* Take $I_k := [-k, k]$ and the seminorms

$$|f|_k := \max_{\ell\leq k} \max_{x\in I_k} |f^{(\ell}(x)| = \max_{\ell\leq k} \|f^{(\ell)}\|_{L^\infty(I_k)}.$$

This is a separating sequence of seminorms: to show it, note that if $|f|_k = 0$, then $f \equiv 0$ in $I_k$ and $\cup_k I_k = \mathbb{R}$. We then build the metric

$$d(f,g) := \sum_{k=1}^\infty \frac{1}{2^k} \frac{|f-g|_k}{1+|f-g|_k}.$$

This construction is abstract and always leads to a metric space where convergence of sequence is equivalent to

$$\lim_n |f_n - f|_k = 0 \qquad \forall k.$$

(b) *Characterization of convergence.* It is clear that uniform convergence on compact sets of $(f_n^{(\ell)})_{n\geq 1}$ for all $\ell$ implies that for given $k$, $|f_n - f|_k \to 0$. (This follows from the fact that only $k+1$ derivatives are involved here and we have restricted our attention to the interval $I_k$. Let now $M$ be a compact set in $\mathbb{R}$ and let $\ell$ be a fixed integer. We then take $k$ such that $\ell \leq k$ and $M \subset [-k,k] = I_k$. Since $|f_n - f|_k \to 0$, then $f_n^{(\ell)} \to f^{(\ell)}$ uniformly in $I_k$ and, therefore, in $M$.

(c) *'Fréchetness.'* Let $(f_n)_{n\geq 1}$ be Cauchy in $X$. The general argument that works for all constructions of metric spaces from sequences of seminorms shows that this implies that for all $\varepsilon > 0$ and $k \geq 0$, there exists $N$ such that

$$|f_n - f_m|_k < \varepsilon \qquad \forall n, m \geq N.$$

In a first instance, we only focus in the function without derivatives.

$$\|f_n - f_m\|_{L^\infty(I_k)} \leq |f_n - f_m|_k < \varepsilon \qquad \forall n, m \geq N.$$

This proves that $f_n$ converges uniformly to a continuous function in $I_k$. If we look at the interval $I_{k+1}$, we prove that $f_n|_{I_{k+1}}$ converges uniformly to a continuous function in $I_{k+1}$. By induction, we have $f_n \to f$ uniformly in $I_k$ for all $k$, where the limit does not depend on $k$. We can now repeat the same argument with all the derivatives.

7. **Diagonal operators in separable Hilbert spaces.** $(4 \times 5$ points) Let $H$ be a complex separable Hilbert space and let $(\phi_n)_{n\geq 1}$ be a Hilbert basis of $H$. Consider a sequence $\lambda := (\lambda_n)_{n\geq 1} \in \ell^\infty$ and the operator defined with the series

$$\Lambda x := \sum_{n=1}^{\infty} \lambda_n (x, \phi_n)\phi_n.$$

(a) Show that $\Lambda$ is bounded and

$$\|\Lambda\| \leq \|\lambda\|_{\ell^\infty}.$$

**Solution.** We can easily estimate (the sum is orthogonal) for all $M > N$ and $x \in X$:

$$\left\|\sum_{n=N+1}^{M} \lambda_n (x, \phi_n)\phi_n\right\|^2 = \sum_{n=N+1}^{M} |\lambda_n|^2 |(x, \phi_n)|^2 \leq \|\lambda\|_{\ell^\infty}^2 \sum_{n=N+1}^{M} |\lambda_n|^2 |(x, \phi_n)|^2.$$

Since, by Parseval's identity,

$$\sum_{n=1}^{N} |(x, \phi_n)|^2 = \|x\|^2$$

it follows that the partial sums $\Lambda_N x := \sum_{n=1}^{N} \lambda_n (x, \phi_n)\phi_n$ are Cauchy and, therefore, convergent. We can then bound

$$\|\Lambda_N x\|^2 = \left\|\sum_{n=1}^{N} \lambda_n (x, \phi_n)\phi_n\right\|^2 = \sum_{n=1}^{N} |\lambda_n|^2 |(x, \phi_n)|^2 \leq \|\lambda\|_{\ell^\infty}^2 \|x\|^2 \qquad \forall x, \quad \forall N$$

and take the limit as $N \to \infty$, showing that $\|\Lambda x\| \leq \|\lambda\|_{\ell^\infty} \|x\|$ for all $x$. Since $\Lambda$ is linear (this is very easy to check), then $\Lambda$ is bounded and $\|\Lambda\| \leq \|\lambda\|_{\ell^\infty}$.

(b) Show that $\Lambda$ is injective if and only if $\lambda_n \neq 0$ for all $n$. (Hint. Compute $\|\Lambda x\|$ and study the kernel of $\Lambda$.)

**Solution.** If $\lambda_n = 0$, then $\Lambda\phi_n = \lambda_n\phi_n = 0$ and $\Lambda$ is not injective. Assume now that $\lambda_n \neq 0$ for all $n$. If $\Lambda x = 0$, then

$$\sum_{n=0}^{\infty} |\lambda_n|^2 |(x, \phi_n)|^2 = 0,$$

which implies that $(x, \phi_n) = 0$ for all $n$. Since $(\phi_n)_{n\geq 1}$ is a Hilbert basis, it follows that $x = 0$ and $\Lambda$ is injective.

(c) Show that $\Lambda$ is invertible if and only if there exists $C > 0$ such that $|\lambda_n| \geq C$. (Hint. For one implication, construct the inverse. For the other one, use the Banach Isomorphism Theorem.)

**Solution.** If $\Lambda$ is invertible, by the Banach Isomorphism Theorem, $\Lambda^{-1}$ is bounded. Therefore

$$\|\Lambda^{-1}x\| \leq D\|x\| \qquad \forall x \in H.$$

Taking $x = \phi_n$ and recalling that $\Lambda\phi_n = \lambda_n\phi_n$, it follows that

$$\frac{1}{|\lambda_n|} \leq D \quad \forall n,$$

so the result follows with $C = 1/D$. Assume now that $|\lambda_n| \geq C > 0$ for all $n$. Then, the sequence $\mu_n := 1/\lambda_n$ is in $\ell^\infty$ and the operator

$$\mathrm{M}x := \sum_{n=1}^{\infty} \mu_n (x, \phi_n)\phi_n$$

is well defined and bounded. Note that

$$(\Lambda x, \phi_n) = \lambda_n (x, \phi_n) \qquad (\mathrm{M}x, \phi_n) = \tfrac{1}{\lambda_n}(x, \phi_n) \qquad \forall n, \qquad \forall x.$$

This and the fact that $x = \sum_n (x, \phi_n)\phi_n$ prove that $\mathrm{M}\Lambda x = \Lambda\mathrm{M}x = x$, for all $x$.

(d) Consider the sequence of operators defined with the partial sums

$$\Lambda_N x := \sum_{n=1}^{N} \lambda_n (x, \phi_n)\phi_n, \qquad N \geq 1.$$

Show that if $\lambda_n \to 0$, then

$$\|\Lambda_N - \Lambda\| \to 0.$$

**Solution.** Applying the first result of this series (question (a)) to the sequence $(0, \ldots, 0, \lambda_{N+1}, \lambda_{N+2}, \ldots)$, it follows that

$$\|\Lambda_N - \Lambda\| \leq \sup_{n\geq N+1} |\lambda_n|.$$

However, if $\lambda_n \to 0$, the right-hand side of the above converges to zero.