Your first day at work – MATH 806 (Fall 2015)

1. Let X be a set (with no particular algebraic structure). A function $d : X \times X \to \mathbb{R}$ is called a metric on X (and then X is called a metric space) when d satisfies the following axioms:

A sequence $(x_n) \subset X$ is said to converge to $x \in X$ when

Sequences in a metric space can only have one limit. The proof is based on this simple inequality

$$d(x, x') \leq d(x, x_n) + d(x', x_n).$$

Finish it.

A sequence (x_n) is said to be a Cauchy sequence (we typically just say the the sequence is Cauchy) when

Every convergent sequence is Cauchy. The proof is based on this simple inequality

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x).$$

Finish it.

When in a metric space X, every Cauchy sequence is convergent, we say that X is complete.

2. Let now V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is said to be a norm when it satisfies the following axioms:

When a vector space is equipped with a norm, it is said to be a normed space. Given a norm on a vector space V, we can easily define a metric/distance on V:

The metric brings along the concepts of convergent and Cauchy sequence. A normed space that is complete is called a **Banach space.** Not every normed space is a Banach space.

Remark. Not every metric space is a normed space. To begin with, a metric space does not neet to have a vector structure, while a normed space does. Even on vector spaces, a metric derived from a norm takes values in the entire $[0, \infty)$, while many metrics take values in bounded intervals.

3. Consider now a vector space V and a function $(\cdot, \cdot) : V \times V \to \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, depending on the field over which V is defined. The bracket is called an inner product when it satisfies the following axioms:

Given an inner product, we can define the associated norm as follows:

Given an inner product and its associated norm, the following inequality, known as the Cauchy-Schwarz inequality, holds:

The proof in the real case is very simple. Fix $u, v \in V$ and consider the map

$$\mathbb{R} \ni t \longmapsto (u + t\,v, u + t\,v) \in \mathbb{R}.$$

It is easy to show that this map is a quadratic polynomial with at most one zero. The CS inequality follows from this simple argument. The proof for the complex case is slightly more involved. Using the Cauchy-Schwarz inequality it is easy to prove that every inner product space is a normed space, i.e., the associated norm is actually a norm. We therefore have concepts of convergent and Cauchy sequences. An inner product space that is complete is called a **Hilbert space**.

Remark. Not every inner product space is a normed space. The norm associated to an inner product satisfies the paralelogram identity:

but not every norm satisfies this identity. Not every inner product space is a Hilbert space.

Quiz # 1 — September 9 – MATH 806 (Fall 2015)

Your name:

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1. (2 points) Define Banach space. (No formulas are needed for this definition.)

2. (2 points) Let H be an inner product space and let $\|x\| = (x,x)^{1/2}$ be the associated norm. State the Cauchy-Schwarz inequality.

3. (3 points) Let X be a metric space. Give two equivalent definitions of what we understand by a compact subset of X.

(Wait to be told before you start with the next page.)

4. (3 points) Let (X, d) be a metric space and let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Assume that there exists a convergent subsequence $(x_{n_k})_{k\geq 1}$ (here $(n_k)_{k\geq 1}$ is an increasing sequence of positive integers). Show that $(x_n)_{n\geq 1}$ converges.

(You can discuss Problem 4 with one classmate.
You are not allowed to go back to Questions 1–3)

Quiz # 2 — September 16 – MATH 806 (Fall 2015)

Your name:

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1. (1 point) Define Hilbert space. (No formulas are needed for this definition.)

2. (1 point) Let H be an inner product space and $\|x\| = (x, x)^{1/2}$, where $(\cdot, \cdot)$ is the inner product in H. Write the Cauchy-Schwarz inequality.

3. (2 points) Let $(x_n)_{n\geq 1}$ be a convergent sequence in an inner product space H. If $x = \lim_{n\to\infty} x_n$, show that

$$(x_n, y) \longrightarrow (x, y) \quad \forall y \in H.$$

4. (3 points) Let H be a Hilbert space and $M \subset H$ be a subspace. Show that the set

$$M^{\perp} := \{x \in H \,:\, (x, m) = 0 \quad \forall m \in M\}$$

is a **closed subspace** of H.

5. (3 points) Define the spaces ℓ^p for $1 \leq p < \infty$ and for $p = \infty$. Define their norms.

6. (2 points) Write down Minkowski's inequality for sequences.

7. (3 points) Show that $\ell^p \subset \ell^\infty$ for every $p \in [1, \infty)$. Find $\mathbf{x} \in \ell^\infty$ such that $\mathbf{x} \notin \ell^p$ for any p.

8. (5 points) Let $K \in L^2(\Omega \times \Omega)$ and consider the operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ given by

$$(\Lambda f)(x) = \int_\Omega K(x,y) f(y) dy.$$

(Here Ω is an open set in $\mathbb{R}^d$.) Show that Λ is bounded and

$$\|\Lambda\| \leq \|K\|_{L^2(\Omega\times\Omega)}.$$

Quiz # 3 — September 23 – MATH 806 (Fall 2015)

Your name:

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1. (3 points) Let $A : X \to Y$ be a bounded linear operator between two normed spaces X and Y. Give three different but equivalent definitions of the operator norm $\|A\|$.

2. (2 points) Let $\|\cdot\|_*$ and $\|\cdot\|_\circ$ be two norms defined in a vector space X. What do we mean when we say that these norms are equivalent?

3. (2 points) The Cauchy-Schwarz inequality states that in an inner product space H,

$$|(x, y)| \leq \|x\|\,\|y\| \qquad \forall x, y \in H,$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product and $\|\cdot\|$ the associated norm. Give necessary and sufficient conditions on x and y so that the inequality is an equality.

4. (2 points) Show that in an inner product space H

$$\|x\| = \sup_{0 \neq y \in H} \frac{|(x,y)|}{\|y\|} \qquad \forall x \in H.$$

Is the supremum a maximum?

5. (2 points) Find an element of ℓ^2 that is not in ℓ^1.

6. (2 points) Let $g \in L^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^d$. Consider the operator

$$L^p(\Omega) \ni f \longmapsto \Lambda f := g\, f \in L^p(\Omega),$$

where $p \in [1, \infty)$. Show that Λ is bounded and $\|\Lambda\| \leq \|g\|_{L^\infty(\Omega)}$. (Note that they are actually equal, but I am not asking you for the proof of equality.)

7. Let H be an infinite dimensional inner product space and let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in H. Such a sequence can always be built using the Gram-Schmidt method applied to a sequence of linearly independent elements of H. (Do no prove this!)

   (a) (2 points) Show that $(\phi_n)_{n\geq 1}$ does not contain Cauchy subsequences.

   (b) (1 points) Use (a) to give a direct proof that in an infinite dimensional inner product space the unit ball is not compact.

8. (3 points) Let $(\phi_n)_{n\geq 1}$ be an orthonormal sequence in a Hilbert space H. Bessel's inequality states that

$$\sum_{n=1}^{\infty} |(x, \phi_n)|^2 \leq \|x\|^2 \qquad \forall x \in H.$$

Use this inequality to show that the series

$$\sum_{n=1}^{\infty} (x, \phi_n)\phi_n$$

converges in H for all $x \in H$. (Hint. Show that the sequence of partial sums is Cauchy.)

Quiz # 4 — September 30 – MATH 806 (Fall 2015)

Your name:

1. (2 points) Let X be a normed space. Define its dual X^* and the norm in X^*.

2. (3 points) Let $A : X \to Y$ and $B : Y \to Z$ be bounded operators. Show that $B\,A : X \to Z$ is bounded and

$$\|B\,A\| \le \|B\|\,\|A\|.$$

Solution. For all $x \in X$,

$$\|(B\,A)\,x\|_Z = \|B(Ax)\|_Z \le \|B\|\,\|A\,x\|_Y \le \|B\|\,\|A\|\,\|x\|_X,$$

where we have used the boundedness inequalities for B and A. This implies the boundedness of $B\,A$ and the inequality $\|B\,A\| \le \|B\|\,\|A\|$.

3. (3 points) Let X be a complex normed space and let V be a subspace of X. What does the extension theorem say? (Note that the full statement says two things about the outcome.)

4. (3 points) Let M be a non-empty subset of an inner product space H. Show that

$$M^\perp := \{x \in H \,:\, (x, m) = 0 \quad \forall m \in M\}$$

is a closed subspace of H.

Solution. If $x, y \in M^\perp$ and $\alpha, \beta \in \mathbb{K}$, then

$$(\alpha\,x + \beta\,y, m) = \alpha\,(x, m) + \beta\,(y, m) = 0 \qquad \forall m \in M,$$

so $\alpha\,x + \beta\,y \in M^\perp$. If $(x_n)_{n\ge1}$ is a sequence in $M^\perp$ and $x = \lim_n x_n$, then

$$0 = (x_n, m) \longrightarrow (x, m) \qquad \forall m \in M,$$

which proves that $x \in M^\perp$. Thus $M^\perp$ is closed.

5. (3 points) Let V be a subspace of a normed space X. Prove that if 0 is an interior point to V, then $V = X$. (Hint. Build a ball around the origin.)

Solution. If 0 is an interior point to V, then there exists ρ such that $B(0, \rho) \subset V$. Let $0 \ne x \in X$. Note that $y = \frac{1}{2\rho\|x\|}x \in B(0, \rho) \subset V$ and therefore $x = 2\rho\|x\|y \in V$.

6. (6 points) Let H be a Hilbert space and $(\phi_n)_{n\ge1}$ be an orthonormal sequence. Let $(\alpha_n)_{n\ge1} \in \ell^2$.

(a) Show that the elements

$$s_N := \sum_{n=1}^{N} \alpha_n \phi_n$$

define a Cauchy sequence in H. (Hint. Compute $\|s_N - s_M\|^2$ when $M > N$.)

Solution. Using the fact that $(\phi_n)_{n\geq 1}$ is an orthonormal sequences

$$\|s_N - s_M\|^2 = \left\| \sum_{n=N+1}^{M} \alpha_n \phi_n \right\|^2 = \sum_{n=N+1}^{M} |\alpha_n|^2 = \sigma_M - \sigma_N = |\sigma_M - \sigma_N|,$$

where $\sigma_N = \sum_{n=1}^{N} |\alpha_n|^2$ is the partial sum of the convergent series defining the square of the ℓ^2 norm of $(\alpha_n)_{n\geq 1}$. This proves the result.

(b) Compute

$$\|s_N\|^2 \qquad \text{and} \qquad \lim_{N\to\infty} \|s_N\|.$$

Solution. Using the same argument

$$\|s_N\|^2 = \sum_{n=1}^{N} |\alpha_n|^2 \longrightarrow \|(\alpha_n)_{n\geq 1}\|^2_{\ell^2}.$$

(c) Use (a) and (b) to prove that the map

$$\ell^2 \ni \alpha = (\alpha_n)_{n\geq 1} \longmapsto T\alpha := \sum_{n=1}^{\infty} \alpha_n \phi_n \in H$$

is an isometry from ℓ^2 to H.

Solution. It is simple to prove that T is linear. We also know that for all $\alpha \in \ell^2$

$$T\alpha = \lim_N s_N$$

and therefore $\|T\alpha\| = \lim \|s_N\| = \|\alpha\|_{\ell^2}$.

The Baire Category Theorem

Let X be a complete metric space. We say that and $V \subset X$ is dense in X when $\overline{V} = X$ or, equivalently, when $V \cap \Omega \neq \emptyset$ for all non-empty open $\Omega \subset X$. Consider now a sequence of open dense subsets $(V_k)_{k\geq 1}$ of X. Take Ω non-empty open and $x_0 \in \Omega$. There exists $r_0 > 0$ (which we can assume to satisfy $r_0 \leq 1$) such that

$$B(x_0, 3r_0) := \{x \in X \,:\, d(x, x_0) < 3r_0\} \subset \Omega.$$

For every $k \geq 1$, we can find $x_k \in X$ and $r_k > 0$ such that

$$B(x_k, 3r_k) \subset V_k \cap B(x_{k-1}, r_{k-1}).$$

Prove it.

> **Solution.** The set V_k is open and dense. Therefore $V_k \cap B(x_{k-1}, r_{k-1})$ is non-empty and we can find $x_k \in V_k \cap B(x_{k-1}, r_{k-1})$. The set is also open (intersection of two open sets) and, therefore, we can find a ball centered in x_k contained in the set. We call the radius $3r_k$.

By construction we can make

$$r_k \leq \frac{r_{k-1}}{3} \leq \ldots \leq \frac{r_0}{3^k} \leq \frac{1}{3^k}.$$

Prove it.

> **Solution.** We can take $3r_{k-1} \leq r_k$ in the k-th step.

Note that $x_k \in B(x_{k-1}, r_{k-1})$ for all k and therefore $d(x_{k+1}, x_k) \leq 1/3^k$. With this it is simple to show that the sequence $(x_k)_{k\geq 1}$ is Cauchy. Prove it.

> **Solution.** For any $n \geq 1$
>
> $$\begin{aligned} d(x_k, x_{k+n}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \ldots + d(x_{k+n-1}, x_{k+n}) \\ &\leq \frac{1}{3^k} + \frac{1}{3^{k+1}} + \ldots + \frac{1}{3^{k+n-1}}. \end{aligned}$$
>
> This shows that $(x_k)_{k\geq 1}$ is Cauchy, since $\sum_{k=0}^{\infty} 1/3^k < \infty$.

Let x^* be the limit of this sequence (X is complete). Then

$$d(x^*, x_k) \leq \sum_{j=k}^{\infty} d(x_{j+1}, x_j) \qquad \forall k.$$

Prove it.

Solution. By the triangle inequality

$$\begin{aligned} d(x^*, x_k) &\leq d(x_k, x_{k+1}) + \ldots + d(x_{k+N-1}, x_{k+N}) + d(x_{k+N}, x^*) \\ &\leq \sum_{j=k}^{\infty} d(x_{j+1}, x_j) + d(x_{k+N}, x^*) \qquad \forall N \geq 0. \end{aligned}$$

Since we can make the last term in the sum as small as we want by taking N large enough, this proves the inequality.

This implies that

$$d(x^*, x_k) \leq \frac{3}{2} r_k \qquad \forall k,$$

and therefore $x^* \in B(x_k, 3r_k) \subset V_k$. Prove it.

Solution. Using the previous inequality and the bounds on the radii

$$\begin{aligned} d(x_k, x^*) &\leq \sum_{j=k}^{\infty} d(x_{j+1}, x_j) \leq \sum_{j=k}^{\infty} r_j \\ &\leq \sum_{\ell=1}^{\infty} \frac{r_k}{3^\ell} = \frac{2}{3} r_k. \end{aligned}$$

Note that we have show that there exists a point x^* such that

$$x^* \in \cap_{k=1}^{\infty} V_k.$$

However, $d(x^*, x_0) \leq \frac{3}{2} r_0$ and therefore $x_0 \in B(x_0, 3r_0) \subset \Omega$. Summing up, for every open set Ω, there exists

$$x^* \in \Omega \cap (\cap_{k=1}^{\infty} V_k).$$

This proves that $\cap_{k=1}^{\infty} V_k$ is dense in X. This completes the proof of the following result:

Baire Category Theorem. *The intersection of a sequence of dense open sets in a complete metric space is dense in the space. In other words, if X is a complete metric space and V_k are open dense subsets of X for all $k \geq 1$, then $\cap_{k=1}^{\infty} V_k$ is dense in X.*

Let now F_n be closed subsets of a complete metric space. Prove that if the interior of F_n is empty for all n, that is, $\mathring{F}_n = \emptyset$, then $\cup_{n=1}^{\infty} F_n \neq X$. (Hint. Use the Baire category theorem with $V_n := X \setminus F_n$. You only need to use that $\cap_{n=1}^{\infty} V_n$ is not empty.)

Solution. Let $V_n := X \setminus F_n$, which is open since F_n is closed. Note that $\overline{V_n} = X \setminus \text{int} F_n$, so the hypotheses on F_n imply that V_n is open and dense in X for all n. Therefore, $\cap_n V_n = X \setminus \cup_n F_n$ is non-empty. This proves the result.

Rephrased in a slightly different (while equivalent) way, we have proved the following well-known consequence of the Baire Category Theorem.

Proposition. *If X is a complete metric space and $X = \cup_{n=1}^{\infty} F_n$ with F_n closed for all n, then there exists n such that $\mathring{F}_n \neq \emptyset$.*

Work day — October 7 – MATH 806 (Fall 2015)

The Banach-Steinhaus Theorem

Today X and Y will be two generic Banach spaces. Let now $\Lambda \in \mathcal{B}(X;Y)$. Prove the inequality

$$\|\Lambda x\|_Y \geq \|\Lambda x_0\|_Y - \|\Lambda\| \, \|x - x_0\|_X \qquad \forall x, x_0 \in X$$

and use it to show that for all $C > 0$ the set

$$S_C := \{x \in X \, : \, \|\Lambda x\|_Y > C\}$$

is open.

Solution. Since $\|\Lambda(x - x_0)\|_Y \leq \|\Lambda\| \, \|x - x_0\|_X$ (by definition of operator norm), the inverse triangle inequality

$$\begin{aligned} \|\Lambda x\|_Y &= \|\Lambda x_0 + \Lambda(x - x_0)\|_Y \\ &\geq \Big| \|\Lambda x_0\|_Y - \|\Lambda(x - x_0)\|_Y \Big| \geq \|\Lambda x_0\|_Y - \|\Lambda(x - x_0)\|_Y \\ &\geq \|\Lambda x_0\|_Y - \|\Lambda\| \, \|x - x_0\|_X \end{aligned}$$

proves the inequality. If $x_0 \in S_C$, then $\|\Lambda x_0\|_Y = C + \varepsilon$ with $\varepsilon > 0$. Let then $\rho := \varepsilon/(2\|\Lambda\|)$. If $\|x - x_0\|_X < \rho$, then

$$\|\Lambda x\|_Y > C + \varepsilon - \|\Lambda\|\rho = C + \varepsilon/2 > C,$$

which proves that $B(x_0; \rho) \subset S_C$ and therefore S_C is open.

Show that if there exists $x_0 \in X$ and $\rho > 0$ such that

$$S_C \cap B(x_0; \rho) = \emptyset, \qquad \text{where } B(x_0; \rho) := \{x \in X \, : \, \|x - x_0\|_X < \rho\}$$

then, using the inequality

$$\|\Lambda x\|_Y \leq \|\Lambda x_0\|_Y + \|\Lambda(x + x_0)\|_Y \qquad \|x\|_X \leq \rho,$$

we can prove that

$$\|\Lambda\| \leq \frac{2C}{\rho}.$$

Solution. By definition of S_C and noting that $S_C \cap B(x_0; \rho) = \emptyset$, it is clear that $\|\Lambda z\|_Y \leq C$ for all $z \in B(x_0; \rho)$. By continuity of Λ it is then clear that $\|\Lambda z\|_Y \leq C$ if $\|z - x_0\|_X \leq \rho$. Therefore,

$$\begin{aligned} \|\Lambda x\|_Y &= \|\Lambda(x + x_0) - \Lambda x_0\| \\ &\leq \|\Lambda x_0\|_Y + \|\Lambda(x + x_0)\|_Y \leq 2C \qquad \|x\|_X \leq \rho, \end{aligned}$$

and scaling

$$\|\Lambda x\|_Y = \frac{1}{\rho}\|\Lambda(\rho x)\| \leq \frac{2C}{\rho} \qquad \text{if } \|x\| = 1.$$

Let now $\mathcal{F} \subset \mathcal{B}(X;Y)$ be an arbitrary collection of bounded linear operators from X to Y. Show that

$$V_n := \{x \,:\, \|\Lambda x\|_Y > n \quad \text{for some } \Lambda \in \mathcal{F}\}$$

is open.
(Hint. Write it as the union of open sets.)

Solution. We can write

$$V_n = \cup_{\Lambda\in\mathcal{F}}\{x \,:\, \|\Lambda x\|_Y > n\}$$

and each of the sets in the right-hand side is open.

Show that if V_n is not dense in X, then

$$\sup_{\Lambda\in\mathcal{F}} \|\Lambda\| < \infty.$$

(Hint. $A \subset X$ is not dense in X if and only if there exists a ball $B(x_0;\rho)$ that does not intersect A.))

Solution. If V_n is not dense in X, there exists a ball $B(x_0;\rho)$ not intersecting V_n. Therefore

$$\{x \,:\, \|\Lambda x\|_Y > n\} \cap B(x_0;\rho) = \emptyset \qquad \forall \Lambda \in \mathcal{F}$$

and by what we proved above

$$\|\Lambda\| \leq \frac{2n}{\rho} \qquad \forall \Lambda \in \mathcal{F}.$$

Use the previous arguments and Baire's Theorem[1] to prove the following result:

Banach-Steinhaus Theorem (a.k.a. Uniform Boundedness Principle)
If X and Y are Banach spaces and $\mathcal{F} \subset \mathcal{B}(X;Y)$, then either

$$\sup_{\Lambda\in\mathcal{F}} \|\Lambda\| < \infty$$

or the set

$$\{x \in X \,:\, \sup_{\Lambda\in\mathcal{F}} \|\Lambda x\|_Y = \infty\}$$

is dense in X.

Solution. Note that

$$\begin{aligned}\{x \,:\, \sup_{\Lambda\in\mathcal{F}} \|\Lambda x\|_Y = \infty\} &= \cap_{n=1}^{\infty}\{x \,:\, \|\Lambda x\|_Y > n \quad \text{for some } \Lambda \in \mathcal{F}\} \\ &= \cap_{n=1}^{\infty} V_n.\end{aligned}$$

We have two options:

[1]The intersection of a sequence of dense open sets in a complete metric space is dense in the space. In other words, if X is a complete metric space and V_n are open dense subsets of X for all $n \geq 1$, then $\cap_{n=1}^{\infty} V_n$ is dense in X.

(a) One of the sets V_n is not dense in X and, therefore, the set $\mathcal{F}$ is bounded in $\mathcal{B}(X;Y)$.

(b) All the sets V_n are dense in X. By Baire's Theorem, the intersection of all of them is dense in X.

Note that the following result is a partial statement of the Banach-Steinhaus Theorem

Let $\mathcal{F} \subset \mathcal{B}(X;Y)$, where X and Y are Banach spaces. If

$$\sup_{\Lambda \in \mathcal{F}} \|\Lambda x\|_Y =: C_x < \infty \qquad \forall x \in X,$$

then the set $\mathcal{F}$ is bounded in the operator norm.

Prove the following corollary now (note that you need to prove linearity and boundedness):

Corollary. *Let $(\Lambda_n)_{n\geq 1}$ be a sequence in $\mathcal{B}(X;Y)$, where X and Y are Banach spaces. If the following limit exists*

$$\lim_{n\to\infty} \Lambda_n x =: \Lambda x \qquad \forall x \in X,$$

then $\Lambda : X \to Y$ is linear and bounded.

Solution. Let $x, x' \in X$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\begin{aligned}
\Lambda(\alpha x + \beta x') &= \lim_{n\to\infty} \Lambda_n(\alpha x + \beta x') \\
&= \lim_{n\to\infty} (\alpha \Lambda_n x + \beta \Lambda_n x') \\
&= \alpha \lim_{n\to\infty} \Lambda_n x + \beta \lim_{n\to\infty} \Lambda_n x' \\
&= \alpha \Lambda x + \beta \Lambda x',
\end{aligned}$$

since Λ_n is linear for all n. Consider now the set $\mathcal{F} := \{\Lambda_n \,:\, n \geq 1\}$. Since $\Lambda_n x$ is convergent, then $\|\Lambda_n x\| \leq C_x$ for all n and all $x \in X$. By the Banach-Steinhaus theorem there exists C such that $\|\Lambda_n\| \leq C$ for all n. Since $\Lambda_n x \to \Lambda x$, then

$$\|\Lambda x\| = \lim_{n\to\infty} \|\Lambda_n x\| \leq \limsup_{n\to\infty} \|\Lambda_n\| \, \|x\| \leq C\|x\|,$$

which proves that Λ is bounded.

Note that the result does not make any claims on whether $\Lambda_n \to \Lambda$ (in the operator norm). The result can be read as: the pointwise limit of a sequence of bounded linear operator is a bounded linear operator.
To show the following result, identify $x \in X$ with $i\,x \in X^{**}$, where

$$(i\,x)(\phi) := \phi(x) \qquad x \in X, \quad \phi \in X^*.$$

Corollary. *Let X be a Banach space and let $x_n \rightharpoonup x$. Then there exists $C > 0$ such that*

$$\|x_n\| \leq C \qquad \forall n.$$

In other words, weakly convergent sequences are bounded.

Solution. Weak convergence means that

$$\phi(x_n) \longrightarrow \phi(x) \qquad \forall \phi \in X^*,$$

or, equivalently,

$$(i\, x_n)(\phi) \longrightarrow (i\, x)(\phi) \quad \forall \phi \in X^*.$$

We then apply the previous result with $\Lambda_n := i\, x_n : X^* \to \mathbb{K}$, to get $\|x_n\|_X = \|i\, x_n\|_{X^{**}} \leq C$ for all n. This proves the result.