NAME:

MATH 835: Evolutionary Partial Differential Equations

Fall'12 | In-class Quiz #2 | October 19

The main problem

We are going to examine carefully some of the more or less painful details concerning weak and strong solutions. The origin of the story is a hyperbolic conservation law

$$u_t + q(u)_x = 0 \qquad \text{or equivalently} \qquad u_t + q'(u)u_x = 0. \tag{1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^2$ and we assume that there exists a continuously differentiable function r such that

$$r(q'(x)) = x \qquad \forall x. \tag{2}$$

The function r is defined in the interval of all possible wave speeds of the model:

$$W := (\inf_{\xi\in\mathbb{R}} q'(\xi), \sup_{\xi\in\mathbb{R}} q'(\xi)) = (w_{\inf}, w_{\sup})$$

To fill the gaps with rarefaction waves starting at the point $(x_0, 0)$, we propose the function

$$u_{\text{fill}}(x,t) := r\left(\frac{x - x_0}{t}\right),$$

defined on the cone $\{(x,t) : \frac{x-x_0}{t} \in W\}$.

(a) (10 points) Show that u_{fill} is a strong solution of the equation (1) wherever it is defined. (This is an easy computation by hand. No need to use complicated arguments.)

(b) (10 points) Take now two constant values $\{u_+, u_-\}$ such that

$$\tfrac{1}{2} = q'(u_-) < q'(u_+) = 1.$$

and consider the function

$$g(x) = \begin{cases} u_-, & \text{if } x < x_0, \\ u_+, & \text{if } x > x_0. \end{cases}$$

Compute all the characteristics generated by this initial condition. Make a plot of the characteristics, with equal scaling in the (x, t) plane (i.e., make the real slopes of the lines visible).

(c) (10 points) **A solution with a rarefaction wave.** Consider the three regions

$$\begin{aligned} O_- &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x < x_0 + \tfrac{t}{2}\}, \\ O_m &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x_0 + \tfrac{t}{2} < x < x_0 + t\}, \\ O_+ &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x > x_0 + t\}, \end{aligned}$$

and the function

$$u(x,t) = \begin{cases} u_-, & \text{in } O_-, \\ u_{\text{fill}}(x,t), & \text{in } O_m, \\ u_+ & \text{in } O_+. \end{cases}$$

Show that u is a solution of (1) in O_-, in O_+ and in O_m. (Note that this has laready been done in O_m.).

(d) (10 points) Show that the function u in (c) is continuous in $\mathbb{R} \times (0,\infty)$. Continuity is clear in each of the three regions. You only need to show it across the two interfaces. For instance, to show continuity in points in the line between O_m and O_+, it suffices (why?) to show that if $(x,t) \in O_m$ and $(x,t) \to (x_0 + \widetilde{t}, \widetilde{t})$, then $u_{\text{fill}}(x,t) \to u_+$.

(e) (10 points + 10 extra points) Take now $v \in \mathcal{D}(\mathbb{R} \times (0, \infty))$. Detail the argument that shows that the function u in (c) satisfies

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} (uv_t + q(u)v_x) \mathrm{d}x \mathrm{d}t = 0. \tag{3}$$

Hint: you can always choose values $A, \varepsilon > 0$ such that the support of v is contained in the rectangle $(x_0 - A, x_0 + A) \times (\varepsilon, A - \varepsilon)$. This will separate the integration domain into three clean cut areas. It'll help you to plot them, give them a name, name the interfaces, plot the normal vectors on the interfaces, etc.

(Additional room for (e))

(f) (10 points + 10 extra points) **A solution with a shock wave.** Consider the number

$$w_0 := \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

Prove that

$$\tfrac{1}{2} < w_0 < 1.$$

(To help you a little bit with this, from condition (2) you can get to show that q'' never vanishes and therefore q' is monotonic.) Show that the function

$$u_{\text{bad}}(x,t) := \begin{cases} u_- & \text{if } x < x_0 + w_0 t, \\ u_+ & \text{if } x > x_0 + w_0 t. \end{cases}$$

satisfies (3) as well.

(Additional room for (f))