Calculus and Analysis Notes for MATH 835

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Theorem. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f_m : \Omega \to \mathbb{R}$ be functions satisfying:

(a) f_m are continuous for all m

(b) there exists constants M_m such that

$$|f_m(\mathbf{x})| \le M_m \qquad \forall \mathbf{x} \in \Omega \qquad \text{with} \qquad \sum_{m=1}^{\infty} M_m < \infty.$$

Then the series

$$f(\mathbf{x}) = \sum_{m=1}^{\infty} f_m(\mathbf{x})$$

converges uniformly in Ω to a continuous function.

Some comments.

1. Properly speaking, the Weierstrass M-test deals with functions $f_m : \Omega \to \mathbb{R}$ defined on any set Ω (it does not need to be an open set in $\mathbb{R}^n$). Assuming condition (b), this result says that the series

$$f(\mathbf{x}) = \sum_{m=1}^{\infty} f_m(\mathbf{x})$$

converges uniformly. Uniform convergence means: for all $\varepsilon > 0$ there exists M_0 such that

$$|f(\mathbf{x}) - \sum_{m=1}^{M} f_m(\mathbf{x})| \le \varepsilon \qquad \forall M \ge M_0, \qquad \forall \mathbf{x} \in \Omega.$$

(The qualifier *uniform* makes reference to the fact that M_0 does not depend on $\mathbf{x}$.) You can find a simple proof of this result in the Wikipedia for instance.

`http://en.wikipedia.org/wiki/Weierstrass_M-test`

2. There is a second part to this result. *If a sequence of continuous functions converges uniformly, then the limit is also continuous.* This is often called the Uniform Limit Theorem. This part of the result is more restrictive in where the theorem is stated. For the sake of what we need, an open set in $\mathbb{R}^n$ will do (but any topological space works too). Note that a result that applies to convergence of sequences, applies to convergence of series, since convergence of the series $\sum_{m=1}^{\infty} f_m$ is defined as convergence of the sequence of partial sums:

$$s_M := \sum_{m=1}^{M} f_m.$$

The very simple proof of this result, can also be found in the Wikipedia

`http://en.wikipedia.org/wiki/Uniform_limit_theorem`

Theorem. Let $\Omega \subset \mathbb{R}^n$ be an open set and let x be one of the components of the vector variable $\mathbf{x} \in \Omega$. Assume that the functions $f_m : \Omega \to \mathbb{R}$ satisfy:

(a) f_m and $\partial_x f_m$ are continuous for all m,

(b) there exist constants M_m such that

$$|f_m(\mathbf{x})| \le M_m \qquad \forall \mathbf{x} \in \Omega \qquad \text{with} \qquad \sum_{m=1}^{\infty} M_m < \infty$$

(c) there exist constants $\widetilde{M}_m$ such that

$$|\partial_x f_m(\mathbf{x})| \le \widetilde{M}_m \qquad \forall \mathbf{x} \in \Omega \qquad \text{with} \qquad \sum_{m=1}^{\infty} \widetilde{M}_m < \infty.$$

Let finally

$$f(\mathbf{x}) = \sum_{m=1}^{\infty} f_m(\mathbf{x}).$$

Then $\partial_x f$ is well defined in Ω, it is continuous, and

$$\partial_x f_m(\mathbf{x}) = \sum_{m=1}^{\infty} \partial_x f_m(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega.$$

Some notes. Most of this misleadingly long statement is a direct application of the Weierstrass M-test and the Uniform Limit Theorem (see the previous page). With those results at hand, we know that the series

$$f(\mathbf{x}) = \sum_{m=1}^{\infty} f_m(\mathbf{x}) \qquad \text{and} \qquad g(\mathbf{x}) = \sum_{m=1}^{\infty} \partial_x f_m(\mathbf{x})$$

converge uniformly to continuous functions. The only detail that's left is showing that $\partial_x f = g$. Since the definition of partial derivative is purely one-dimensional (to define a partial derivative with respect to a variable, all other variables are frozen), we only need to prove the result for functions of one variable, that is, for functions f_m of only one variable, we want to prove that

$$\lim_{M\to\infty} \Big(\sum_{m=1}^{M} f_m'(t) \Big) = \frac{\mathrm{d}}{\mathrm{d}t} \Big(\sum_{m=1}^{\infty} f_m(t) \Big).$$

Fixing the value of t, what we have to do is showing that

$$\sum_{m=1}^{\infty} \Big(\lim_{h\to 0} \frac{f_m(t+h) - f_m(t)}{h} \Big) = \lim_{h\to 0} \Big(\sum_{m=1}^{\infty} \frac{f_m(t+h) - f_m(t)}{h} \Big),$$

which is, once more, a consequence of the M-test, applied now to the functions

$$g_m(h) := \begin{cases} \frac{1}{h}(f_m(t+h) - f_m(t)), & h \neq 0, \\ f_m'(t), & h = 0. \end{cases}$$

Theorem. Let $f : \Omega \times (0,T) \to \mathbb{R}$ be continuous and differentiable in the last variable and assume that

$$|f(\mathbf{x},t)| + |f_t(\mathbf{x},t)| \leq g(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega, \forall t \in (0,T),$$

where

$$\int_\Omega g(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_\Omega f_t(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

There are many different forms of Parseval's Identity. The most usual one is related to the full (sines and cosines) real Fourier series. Given $f : (-L, L) \to \mathbb{R}$, such that

$$\int_{-L}^{L} |f(x)|^2 \,\mathrm{d}\, x, \tag{1}$$

we define its Fourier coefficients

$$\begin{aligned} a_m &:= \frac{1}{L}\int_{-L}^{L} f(x)\cos(m\pi\, x)\mathrm{d}x \qquad m \geq 0, \\ b_m &:= \frac{1}{L}\int_{-L}^{L} f(x)\sin(m\pi\, x)\mathrm{d}x \qquad m \geq 1. \end{aligned}$$

Then, Parseval's Identity asserts that

$$\frac{a_0^2}{2} + \sum_{m=1}^{\infty}(a_m^2 + b_m^2) = \frac{1}{L}\int_{-L}^{L}|f(x)|^2\mathrm{d}x.$$

The Fourier series of a square integrable function (a function satisfying (1)) converges in square mean:

$$\lim_{M\to\infty}\int_{-L}^{L}\Big|\frac{a_0}{2} + \sum_{m=1}^{M}(a_m\cos(m\pi\, x) + b_m\sin(m\pi\, x) - f(x)\Big|^2\mathrm{d}x = 0.$$

Cosine series. Foruier cosine series are (in a way) a particular case of Fourier series. We start by considering a function $g : (0, L) \to \mathbb{R}$ such that

$$\int_{0}^{L}|g(x)|^2\mathrm{d}x. \tag{2}$$

We definer its even extension $f_{\mathrm{even}} : (-L, L) \to \mathbb{R}$

$$f_{\mathrm{even}}(x) := \begin{cases} g(x), & x \geq 0, \\ g(-x), & x < 0. \end{cases}$$

The Fourier coefficients of f_{even} can be written in terms of g as follows

$$a_m = \frac{2}{L}\int_{0}^{L} g(x)\sin(m\pi\, x)\mathrm{d}x, \qquad b_m = 0,$$

(the coefficients a_m are the cosine Fourier coefficients of g) and Parseval's Identity becomes

$$\frac{a_0^2}{2} + \sum_{m=1}^{\infty} a_m^2 = \frac{2}{L}\int_{0}^{L}|g(x)|^2\mathrm{d}x.$$

Sine series. If we start again if $g : (0, L) \to \mathbb{R}$ satisfying (2), and create its odd extension

$$f_{\mathrm{odd}}(x) := \begin{cases} g(x), & x \geq 0, \\ -g(-x), & x < 0, \end{cases}$$

the Fourier coefficients are now

$$a_m = 0, \qquad b_m = \frac{2}{L}\int_0^L g(x)\sin(m\pi\, x)\mathrm{d}x$$

(the b_m coefficients are the sine Fourier coefficients of g), and Parseval's Identity is

$$\sum_{m=1}^{\infty} b_m^2 = \frac{2}{L}\int_0^L |g(x)|^2 \mathrm{d}x.$$

For more about Fourier series, read Appendix A of the textbook.

Let us assume that the bounded open domain $\Omega \subset \mathbb{R}^n$ is such that we can define the outer normal vector field $\boldsymbol{\nu}$ almost everywhere on its boundary $\partial\Omega$ and we can integrate on this boundary. Notation for integration will be the same as in the textbook, namely

$$\int_\Omega f \, \mathrm{d}\mathbf{x}$$

is a Lebesgue (volume) integral of the function f (the variable is not made explicit) and $\mathrm{d}\mathbf{x}$ is just the Lebesgue measure. For integrals on $\partial\Omega$, we will write

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma.$$

The simplest version of the divergence theorem says that if $\mathbf{p} : \Omega \to \mathbb{R}^n$ is $\mathcal{C}^1(\overline{\Omega})$ component by component, then

$$\int_\Omega \operatorname{div} \mathbf{p} \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{p} \cdot \boldsymbol{\nu} \, \mathrm{d}\sigma. \tag{3}$$

If we apply this result to a vector field $\mathbf{p} = u\,\mathbf{q}$, where $u \in \mathcal{C}^1(\overline{\Omega})$, $\mathbf{q} \in \mathcal{C}^1(\overline{\Omega})^n$, we obtain another popular form of the result

$$\int_\Omega u \operatorname{div} \mathbf{q} \, \mathrm{d}\mathbf{x} + \int_\Omega \nabla u \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} u\, \mathbf{q} \cdot \boldsymbol{\nu} \, \mathrm{d}\sigma. \tag{4}$$

If we take $u \equiv 1$ and $\mathbf{q} = \mathbf{p}$ in (4), we obtain (3). If we take $\mathbf{q} = \nabla v$ (with $v \in \mathcal{C}^2(\overline{\Omega})$, then we obtain a third form of the divergence theorem (often called Green's First identity)

$$\int_\Omega u\, \Delta v \, \mathrm{d}\mathbf{x} + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} u\, \partial_\nu v \, \mathrm{d}\sigma. \tag{5}$$